Math 3150-1, Practice Midterm Exam 1

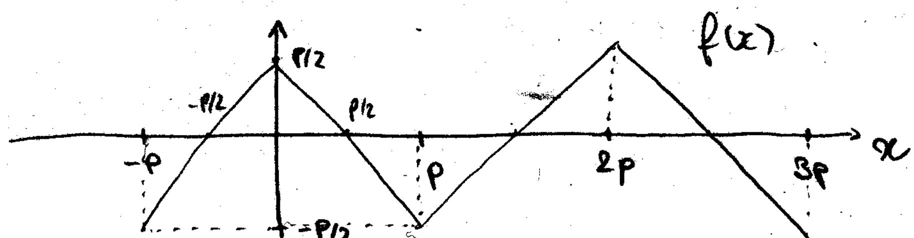
September 18 2008

Total points: 100/100.

Problem 1 (20 pts) Consider the $2p$–periodic function

$$f(x) = \begin{cases} -\left(x - \frac{p}{2}\right) & \text{if } 0 \le x \le p \\ x + \frac{p}{2} & \text{if } -p \le x \le 0 \end{cases}$$

(a) Sketch $f(x)$ for $x \in [-p, 3p]$. Carefully label your axis.



(b) Is f continuous? Piecewise continuous? Piecewise smooth?

f is continuous $\Rightarrow$ f is pcws continuous.

$$f'(x) = \begin{cases} -1 & 0 < x < p \\ +1 & -p < x < 0 \end{cases}, \text{ 2-periodic function} \Rightarrow f' \text{ is pcws cont} \Rightarrow \underline{f \text{ is pcws smooth}}.$$

(c) Find the Fourier series of f.

Since f is an even function ($f(-x) = f(x)$), $\boxed{b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx = 0}$

$$a_n = -\frac{2}{p}\int_0^p (x - p/2)\cos\frac{n\pi}{p}x \qquad (n \ge 1)$$

$$= \cancel{-\frac{2}{p}(x - p/2)\frac{p}{n\pi}\sin\frac{n\pi}{p}x\Big|_0^p} + \frac{2}{p}\frac{p}{n\pi}\int_0^p \sin\frac{n\pi}{p}x\,dx$$

$$= -\frac{2}{n\pi}\left(\frac{p}{n\pi}\right)\cos\frac{n\pi}{p}x\Big|_{x=0}^{p} = -\frac{2p}{(n\pi)^2}\left((-1)^n - 1\right) = \begin{cases} \frac{4p}{(n\pi)^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$a_0 = \frac{1}{p}\int_0^p f(x)\,dx = 0.$$

$$\Rightarrow \boxed{f(x) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x = \sum_{k=0}^{\infty} \frac{4p}{(2k+1)^2\pi^2}\cos\frac{(2k+1)\pi}{p}x}$$

Problem 2 (15 pts) Decide whether the following partial differential equations are linear or non-linear and if linear, whether they are homogeneous or non-homogeneous. Determine the order of the differential equation.

(a) $\begin{cases} u_{xxxx} + u_{xx} = u_{tt} \\ u(0,t) = u_x(0,t) = 0 \\ u(1,t) = u_x(1,t) = 0 \end{cases}$ order: 4, linear & homog

(b) $u_t = uu_x + u_{xx}$ nonlinear, order: 2

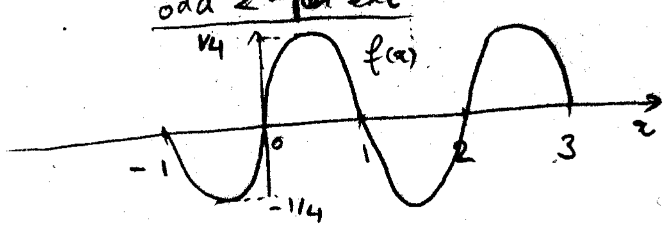
(c) $u_{xx} + u_{yy} = \exp[-x^2 - y^2]$ linear non homog, order: 2

(d) $\begin{cases} u_{xx} = u_t \\ u(0,t) = 0 \\ u_x(1,t) + u(1,t) = 0 \end{cases}$ linear, homog, order: 2

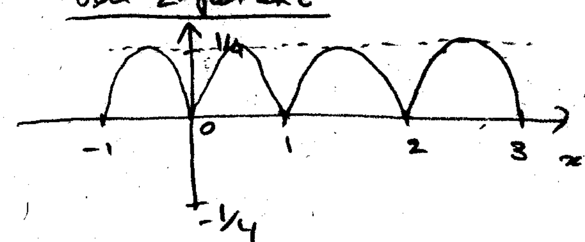
Problem 3 (25 pts) Let $f(x) = x(1-x)$ be defined on $[0,1]$.

(a) Sketch the odd and even 2−periodic extensions of $f(x)$. Carefully label your axis.

odd 2-per ext



even 2-per ext



(b) Calculate the Sine and Cosine Series expansions of $f(x)$.

Sine series: $b_n = \frac{2}{1}\int_0^1 x(1-x)\sin(n\pi x)$

we use the identities (see back of the book. Similar identities will be provided in the exam)

$$(1) \quad \int x \sin ax \, dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax + C$$

$$(2) \quad \int x^2 \sin ax \, dx = \frac{2x}{a^2}\sin ax - \frac{x^2a^2 - 2}{a^3}\cos ax + C$$

$$(1)-(2) \Rightarrow \int x(1-x)\sin ax \, dx = \frac{1-2x}{a^2}\sin ax + \frac{a^2x(x-1)-2}{a^3}\cos ax + C$$

$$\Rightarrow \int_0^1 x(1-x)\sin n\pi x \, dx = \cancel{(\ )\sin n\pi x \Big|_0^1} + \frac{(n\pi)^2 x(x-1) - 2}{(n\pi)^3}\cos(n\pi x)\Big|_0^1$$

0

$$= -\frac{2}{(n\pi)^3}((-1)^n - 1) = \begin{cases} \frac{4}{(n\pi)^3} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Thus:

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{8}{((2k+1)\pi)^3}\sin((2k+1)\pi x)}$$

For the cosine series we proceed similarly:

$$\int x\cos ax\,dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax + C$$

$$\int x^2\cos ax\,dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2-2}{a^3}\sin ax + C$$

$$\Rightarrow \int x(1-x)\cos ax\,dx = \frac{1-2x}{a^2}\cos ax + \frac{a^2x(x-1)-2}{(n\pi)^3}\sin ax + C$$

$$\Rightarrow \int_0^1 x(1-x)\cos n\pi x\,dx = \frac{1-2x}{(n\pi)^2}\cos n\pi x + (\ )\cancel{\sin n\pi x}\Big|_0^1$$

$$= \frac{-((-1)^n+1)}{(n\pi)^2} = \begin{cases} 2/(n\pi)^2 & n \text{ even} \\ 0 & n \text{ odd}\end{cases} \quad (n \geq 1)$$

$$\boxed{a_0 = \int_0^1 x(1-x)\,dx = \frac{x^2}{2}-\frac{x^3}{3}\Big|_0^1 = \frac{1}{2}-\frac{1}{3}=\frac{1}{6}}$$

$$\Rightarrow \boxed{f(x) = \frac{1}{6} + \sum_{k=1}^{\infty}\frac{2}{(2k\pi)^2}\cos(2k\pi x)}$$

(c) Solve the one dimensional wave equation

$$\begin{cases} u_{tt} = 2u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x \cos 2n\pi t}\ ; \text{ with } b_n \text{ computed as in sine series of } f$$

$$= \boxed{\sum_{k=0}^{\infty}\frac{8}{((2k+1)\pi)^3}\sin((2k+1)\pi x)\cos(2(2k+1)\pi t)}$$

(d) Solve the one dimensional wave equation

$$\begin{cases} u_{tt} = 2u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0 \\ u_t(x,0) = f(x) \end{cases}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n^* \sin n\pi x \sin 2n\pi t}\ ; \quad u_t(x,t) = \sum_{n=1}^{\infty} b_n^*(2n\pi)\sin(n\pi x)\cos(2n\pi t)$$

$$f(x) = u_t(x,0) = \sum_{n=1}^{\infty} \underbrace{b_n^*(2n\pi)}_{=b_n \text{ as in (3b)}}\sin(n\pi x)$$

$$\text{where } b_n^* = \begin{cases} \frac{2}{(n\pi)^4} & n \text{ odd} \\ 0 & n \text{ even}\end{cases} \quad \Rightarrow \boxed{u(x,t) = \sum_{k=0}^{\infty}\frac{2}{((2k+1)\pi)^4}\sin((2k+1)\pi x)\sin(2(2k+1)\pi t)}$$

Problem 4 (20 pts) Consider the one dimensional wave equation

$$\begin{cases} u_{tt} = u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0 \\ u_t(x,0) = \cos(\pi x) \end{cases}$$

Solve this differential equation using

(a) Fourier series → We need to first compute the Sine series of $\cos\pi x$.

$$\boxed{b_n} = 2\int_0^1 \cos\pi x \sin n\pi x\,dx = \int_0^1 [\sin(n+1)\pi x - \sin(n-1)\pi x]\,dx \quad (n>1)$$

$$= \frac{1}{(n+1)\pi}\cos(n+1)\pi x\Big|_0^1 - \frac{1}{(n-1)\pi}\cos(n-1)\pi x\Big|_0^1$$

$$= \frac{1}{\pi}\left[\frac{((-1)^{n+1}-1)}{n+1} - \frac{((-1)^{n+1}-1)}{n-1}\right] = \frac{(-1)^{n+1}-1}{\pi}\left(\frac{-2}{n^2-1}\right)$$

$$\boxed{= \begin{cases} \dfrac{4}{\pi(n^2-1)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}}$$

$$b_1 = 2\int_0^1 \cos\pi x \sin\pi x\,dx = \int_0^1 \sin 2\pi x\,dx = \cos 2\pi x\Big|_0^1 = 0.$$

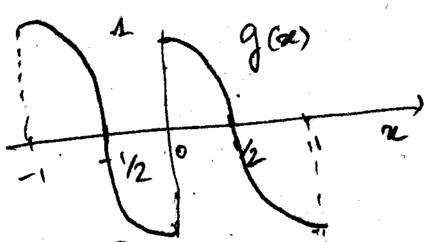
$$u(x,t) = \sum_{n=1}^{\infty} b_n^* \sin n\pi x \sin n\pi t$$

where $b_n^* = \dfrac{b_n}{n\pi}$ (as in problem 3d)

(b) D'Alembert's method

We have: $u(x,t) = \frac{1}{2}\left[G(x+t) - G(x-t)\right]$

where $G(x) = \int g^*(s)\,ds$ and g^* = odd 2-per ext of g:



$-\cos\pi x$ $\quad$ $\cos\pi x$

Thus: $\boxed{G(x)} = \begin{cases} -\int_{-1}^{x}\cos\pi t\,dt & -1<x<0 \\ -\int_{-1}^{0}\cos\pi t\,dt + \int_0^x \cos\pi t\,dt, & 0<x<1 \end{cases}$

$$= \begin{cases} -\frac{1}{\pi}\sin\pi x, & -1<x<0 \\ \frac{1}{\pi}\sin\pi x, & 0<x<1 \end{cases}$$

$$\boxed{= \frac{1}{\pi}|\sin\pi x|.}$$

Problem 5 (20 pts) Consider the following one dimensional heat equations:

$$(A)\begin{cases} u_t = u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = x(1-x). \end{cases} \qquad (B)\begin{cases} u_t = u_{xx} \\ u(0,t) = 1, u(1,t) = 0 \\ u(x,0) = 1 - x^2. \end{cases}$$

(a) Give a physical interpretation of the boundary conditions for (A) and (B).

(A): ice bath on both ends of rod

(B): temp is maintained at $1°C$ at $x=0$

" " " " $0°C$ at $x=1$

(b) Solve problem (A). **Hint:** You may use the results from problem 3c.

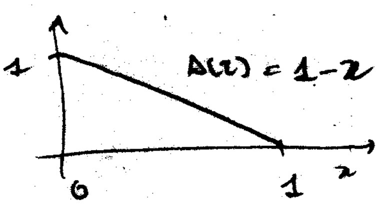
In (3c) we computed b_n s.t. $x(1-x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$,

thus:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x \exp\left[-(n\pi)^2 t\right]}$$

with b_n as in 3c.

(c) What is the steady state temperature distribution $s(x)$ for (B)?



$s(x) = 1 - x$ ← using B.C.

(d) Use your answer to (a) together with $s(x)$ to solve problem (B).

Let w solve (B) then $v(x,t) = w(x,t) - s(x)$ solves:

$$\begin{cases} v_t = v_{xx} \\ v(0,t) = v(1,t) = 0 \\ v(x,0) = x(1-x) \end{cases} \Leftrightarrow (A) \Rightarrow \boxed{v(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \exp\left[-(n\pi)^2 t\right]}$$

where b_n as in (b).

$$\text{Then } \boxed{u(x,t) = v(x,t) + s(x) = 1 - x + \sum_{n\geq 1}^{\infty} b_n \sin(n\pi x) \exp\left[-(n\pi)^2 t\right]}$$