

## Laplace Theory Examples

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**1 Example (Harmonic oscillator)** Solve the initial value problem  $x'' + x = 0$ ,  $x(0) = 0$ ,  $x'(0) = 1$  by Laplace's method.

**Solution:**

The solution is  $x(t) = \sin t$ . The details:

$$L(x'') + L(x) = L(0)$$

$$sL(x') - x'(0) + L(x) = 0$$

$$s[sL(x) - x(0)] - x'(0) + L(x) = 0$$

$$(s^2 + 1)L(x) = 1$$

$$L(x) = \frac{1}{s^2 + 1}$$
$$= L(\sin t)$$

$$x(t) = \sin t$$

Apply  $L$  across the equation.

Parts rule.

Parts rule again.

Use  $x(0) = 0$ ,  $x'(0) = 1$ .

Isolate  $L(x)$  left.

Backward Laplace table.

Lerch's cancelation law.

**2 Example ( $s$ -Differentiation Rule)** Show the steps for  $L(t^2 e^{5t}) = \frac{2}{(s - 5)^3}$ .

**Solution:**

$$\begin{aligned}L(t^2 e^{5t}) &= \left(-\frac{d}{ds}\right) \left(-\frac{d}{ds}\right) L(e^{5t}) \\&= (-1)^2 \frac{d}{ds} \frac{d}{ds} \left(\frac{1}{s - 5}\right) \\&= \frac{d}{ds} \left(\frac{-1}{(s - 5)^2}\right) \\&= \frac{2}{(s - 5)^3}\end{aligned}$$

Use  $s$ -differentiation.

Forward Laplace table.

Calculus power rule.

Identity verified.

**3 Example (First shifting rule)** Show the steps for  $L(t^2 e^{-3t}) = \frac{2}{(s + 3)^3}$ .

**Solution:**

$$\begin{aligned} L(t^2 e^{-3t}) &= L(t^2) \Big|_{s \rightarrow s - (-3)} \\ &= \left( \frac{2}{s^2 + 1} \right) \Big|_{s \rightarrow s - (-3)} \\ &= \frac{2}{(s + 3)^3} \end{aligned}$$

First shifting rule.

Forward Laplace table.

Identity verified.

**4 Example (Trigonometric formulas)** Show the steps used to obtain these Laplace identities:

$$(a) L(t \cos at) = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

$$(c) L(t^2 \cos at) = \frac{2(s^3 - 3sa^2)}{(s^2 + a^2)^3}$$

$$(b) L(t \sin at) = \frac{2sa}{(s^2 + a^2)^2}$$

$$(d) L(t^2 \sin at) = \frac{6s^2a - a^3}{(s^2 + a^2)^3}$$

**Solution:** The details for (b) and (d) are left as exercises.

The details for (a):

$$\begin{aligned} L(t \cos at) &= -(d/ds)L(\cos at) \\ &= -\frac{d}{ds} \left( \frac{s}{s^2 + a^2} \right) \\ &= \frac{s^2 - a^2}{(s^2 + a^2)^2} \end{aligned}$$

Use  $s$ -differentiation.

Forward Laplace table.

Calculus quotient rule.

The details for (c):

$$\begin{aligned} L(t^2 \cos at) &= -(d/ds)L((-t) \cos at) \\ &= \frac{d}{ds} \left( -\frac{s^2 - a^2}{(s^2 + a^2)^2} \right) \\ &= \frac{2s^3 - 6sa^2}{(s^2 + a^2)^3} \end{aligned}$$

Use  $s$ -differentiation.

Result of (a).

Calculus quotient rule.

**5 Example (Exponentials)** Show the steps used to obtain these Laplace identities:

$$(a) L(e^{at} \cos bt) = \frac{s - a}{(s - a)^2 + b^2} \quad (c) L(te^{at} \cos bt) = \frac{(s - a)^2 - b^2}{((s - a)^2 + b^2)^2}$$

$$(b) L(e^{at} \sin bt) = \frac{b}{(s - a)^2 + b^2} \quad (d) L(te^{at} \sin bt) = \frac{2b(s - a)}{((s - a)^2 + b^2)^2}$$

**Solution:** Left as exercises are (a), (b) and (d).

Details for (c):

$$\begin{aligned} L(te^{at} \cos bt) &= L(t \cos bt)|_{s \rightarrow s-a} \\ &= \left( -\frac{d}{ds} L(\cos bt) \right) \Big|_{s \rightarrow s-a} \\ &= \left( -\frac{d}{ds} \left( \frac{s}{s^2 + b^2} \right) \right) \Big|_{s \rightarrow s-a} \\ &= \left( \frac{s^2 - b^2}{(s^2 + b^2)^2} \right) \Big|_{s \rightarrow s-a} \\ &= \frac{(s - a)^2 - b^2}{((s - a)^2 + b^2)^2} \end{aligned}$$

First shifting rule.

Use  $s$ -differentiation.

Forward Laplace table.

Calculus quotient rule.

Verified (c).

**6 Example (Hyperbolic functions)** Establish these Laplace transform facts about  $\cosh u = (e^u + e^{-u})/2$  and  $\sinh u = (e^u - e^{-u})/2$ .

$$(a) L(\cosh at) = \frac{s}{s^2 - a^2}$$

$$(b) L(\sinh at) = \frac{a}{s^2 - a^2}$$

$$(c) L(t \cosh at) = \frac{s^2 + a^2}{(s^2 - a^2)^2}$$

$$(d) L(t \sinh at) = \frac{2as}{(s^2 - a^2)^2}$$

**Solution:** Left as exercises are (b) and (c).

The details for (a):

$$\begin{aligned} L(\cosh at) &= \frac{1}{2}(L(e^{at}) + L(e^{-at})) \\ &= \frac{1}{2} \left( \frac{1}{s-a} + \frac{1}{s+a} \right) \\ &= \frac{s}{s^2 - a^2} \end{aligned}$$

Definition and linearity.

Forward Laplace table.

Identity (a) verified.

The details for (d):

$$\begin{aligned} L(t \sinh at) &= -\frac{d}{ds} \left( \frac{a}{s^2 - a^2} \right) \\ &= \frac{a(2s)}{(s^2 - a^2)^2} \end{aligned}$$

Apply the  $s$ -differentiation rule.

Calculus power rule; (d) verified.

**7 Example ( $s$ -Differentiation Rule)** Solve  $L(f(t)) = \frac{2s}{(s^2 + 1)^2}$  for  $f(t)$ .

**Solution:** The solution is  $f(t) = t \sin t$ .

The details:

$$\begin{aligned}L(f(t)) &= \frac{2s}{(s^2 + 1)^2} \\&= -\frac{d}{ds} \left( \frac{1}{s^2 + 1} \right) \\&= -\frac{d}{ds} (L(\sin t)) \\&= L(t \sin t)\end{aligned}$$

$$f(t) = t \sin t$$

Calculus power rule  $(u^n)' = nu^{n-1}u'$ .

Backward Laplace table.

Apply the  $s$ -differentiation rule.

Lerch's cancelation law.



**8 Example (First Shifting Rule I)** Solve  $L(f(t)) = \frac{s + 7}{s^2 + 4s + 8}$  for  $f(t)$ .

**Solution:**

The answer is  $f(t) = e^{-2t}(\cos 2t + \frac{5}{2} \sin 2t)$ . The details:

$$L(f(t)) = \frac{s + 7}{(s + 2)^2 + 4}$$

Complete the square.

$$= \frac{S + 5}{S^2 + 4}$$

Replace  $s + 2$  by  $S$ .

$$= \frac{S}{S^2 + 4} + \frac{5}{2} \frac{2}{S^2 + 4}$$

Split into table entries.

$$= \frac{s}{s^2 + 4} + \frac{5}{2} \frac{2}{s^2 + 4} \Big|_{s \rightarrow S=s+2}$$

Shifting rule preparation.

$$= L\left(\cos 2t + \frac{5}{2} \sin 2t\right) \Big|_{s \rightarrow S=s+2}$$

Forward Laplace table.

$$= L(e^{-2t}(\cos 2t + \frac{5}{2} \sin 2t))$$

First shifting rule.

$$f(t) = e^{-2t}(\cos 2t + \frac{5}{2} \sin 2t)$$

Lerch's cancelation law.

**9 Example (First Shifting Rule II)** Solve  $L(f(t)) = \frac{s + 2}{s^2 + 2s + 2}$  for  $f(t)$ .

**Solution:**

The answer is  $f(t) = e^{-t} \cos t + e^{-t} \sin t$ . The details:

$$L(f(t)) = \frac{s + 2}{s^2 + 2s + 2}$$

$$= \frac{s + 2}{(s + 1)^2 + 1}$$

$$= \frac{S + 1}{S^2 + 1}$$

$$= \frac{S}{S^2 + 1} + \frac{1}{S^2 + 1}$$

$$= L(\cos t) + L(\sin t)|_{s \rightarrow S=s+1}$$

$$= L(e^{-t} \cos t) + L(e^{-t} \sin t)$$

$$f(t) = e^{-t} \cos t + e^{-t} \sin t$$

Signal for this method: the denominator has complex roots.

Complete the square, denominator.

Substitute  $S$  for  $s + 1$ .

Split into Laplace table entries.

Forward Laplace table.

First shift rule.

Invoke Lerch's cancelation law.

**10 Example (Damped oscillator)** Solve by Laplace's method the initial value problem  $x'' + 2x' + 2x = 0$ ,  $x(0) = 1$ ,  $x'(0) = -1$ .

**Solution:**

The solution is  $x(t) = e^{-t} \cos t$ . The details:

$$L(x'') + 2L(x') + 2L(x) = L(0)$$

$$sL(x') - x'(0) + 2L(x') + 2L(x) = 0$$

$$s[sL(x) - x(0)] - x'(0) + 2[L(x) - x(0)] + 2L(x) = 0$$

$$(s^2 + 2s + 2)L(x) = 1 + s$$

$$L(x) = \frac{s + 1}{s^2 + 2s + 2}$$

$$= \frac{s + 1}{(s + 1)^2 + 1}$$

$$= L(\cos t)|_{s \rightarrow s+1}$$

$$= L(e^{-t} \cos t)$$

$$x(t) = e^{-t} \cos t$$

Apply  $L$  across the equation.

The  $t$ -derivative rule on  $x'$ .

The  $t$ -derivative rule on  $x$ .

Use  $x(0) = 1$ ,  $x'(0) = -1$ .

Divide to isolate  $L(x)$ .

Complete the square in the denominator.

Forward Laplace table.

First shifting rule.

Lerch's cancelation law.

**11 Example (Second Shifting Rule I)** Show the steps for

$$L(\sin t H(t - \pi)) = \frac{-e^{-\pi s}}{s^2 + 1}.$$

**Solution:**

The second shifting rule is applied as follows.

$$\begin{aligned} \text{LHS} &= L(\sin t H(t - \pi)) \\ &= L(g(t)H(t - a)) \\ &= e^{-as} L(g(t + a)) \\ &= e^{-\pi s} L(\sin(t + \pi)) \\ &= e^{-\pi s} L(-\sin t) \end{aligned}$$

$$\begin{aligned} &= e^{-\pi s} \frac{-1}{s^2 + 1} \\ &= \text{RHS} \end{aligned}$$

Left side of the identity.

Choose  $g(t) = \sin t$ ,  $a = \pi$ .

Second form, second shifting theorem.

Substitute  $a = \pi$ .

Sum rule  $\sin(a + b) = \sin a \cos b + \sin b \cos a$  plus  $\sin \pi = 0$ ,  $\cos \pi = -1$ .

Forward Laplace table.

Identity verified.

**12 Example (Second Shifting Rule II)** Solve  $L(f(t)) = e^{-3s} \frac{s+1}{s^2+2s+2}$  for  $f(t)$ .

**Solution:**

The answer is  $f(t) = e^{3-t} \cos(t-3)H(t-3)$ . The details:

$$L(f(t)) = e^{-3s} \frac{s+1}{(s+1)^2+1}$$

$$= e^{-3s} \frac{S}{S^2+1}$$

$$= e^{-3S+3} (L(\cos t))|_{s \rightarrow S=s+1}$$

$$= e^3 (e^{-3s} L(\cos t))|_{s \rightarrow S=s+1}$$

$$= e^3 (L(\cos(t-3)H(t-3)))|_{s \rightarrow S=s+1}$$

$$= e^3 L(e^{-t} \cos(t-3)H(t-3))$$

$$f(t) = e^{3-t} \cos(t-3)H(t-3)$$

Complete the square.

Replace  $s+1$  by  $S$ .

Forward Laplace table.

Regroup factor  $e^{-3S}$ .

Second shifting rule.

First shifting rule.

Lerch's cancelation law.