The Jules Verne Problem

In his 1865 novel *From the Earth to the Moon*, Jules Verne asked what initial velocity must be given to a projectile in order to reach the moon.

The answer was given in his book, in an elaborate story, which identified the launch point to be near Cape Canaveral, Florida. Some 100 years later, the *Columbia* spacecraft took Verne's journey to the moon. There are remarkable similarities between the US space program and the details in Verne's book.

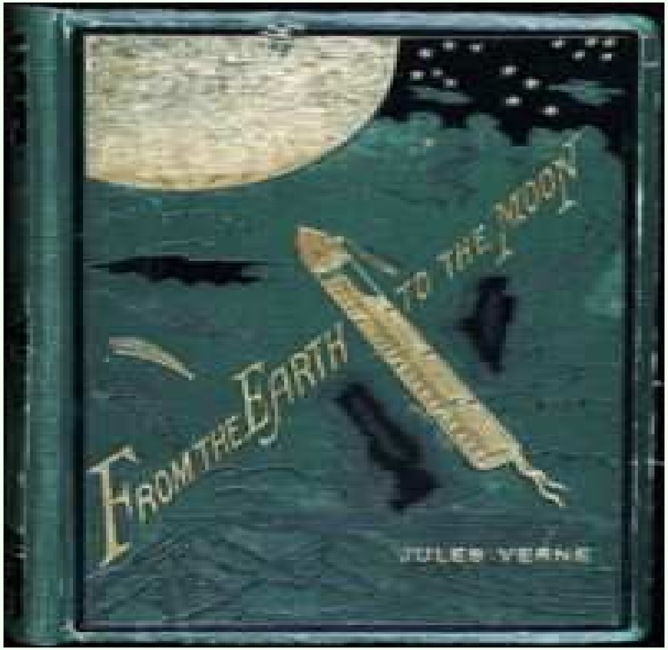

Figure 1. Jules Verne's novel of 1865.

Earth to the Moon

A projectile launched from the surface of the earth is attracted both by the earth and the moon. The altitude $r(t)$ of the projectile above the earth is known to satisfy the initial value problem

$$
\begin{aligned}
&r''(t) = -\frac{Gm_1}{(R_1 + r(t))^2} + \frac{Gm_2}{(R_2 - R_1 - r(t))^2}, \\
&r(0) = 0, \quad r'(0) = v_0.
\end{aligned} \tag{1}
$$

Earth to the Moon Constants

The unknown initial launch velocity v_0 of the projectile is given in meters per second. The constants in the Jules Verne problem are determined as follows.

$G = 6.6726 \times 10^{-11}$ N-m^2/kg^2	Universal gravitation constant,
$m_1 = 5.975 \times 10^{24}$ kilograms	Mass of the earth,
$m_2 = 7.36 \times 10^{22}$ kilograms	Mass of the moon,
$R_1 = 6,378,000$ meters	Radius of the earth,
$R_2 = 384,400,000$ meters	Distance from the earth's center to the moon's center.

Reformulated Jules Verne Problem

Verne's question in terms of equation (1) becomes:

> What minimal value of v_0 causes the projectile to have zero net acceleration at some point between the earth and the moon?

If such a *point of no return* occurs at distance $r = r^*$, then the projectile must fall to the moon, instead of back to earth. Let $r''(t) = 0$ in (1) and substitute r^* for $r(t)$ in the resulting equation. Then

$$
\begin{aligned}
&-\frac{Gm_1}{(R_1 + r^*)^2} + \frac{Gm_2}{(R_2 - R_1 - r^*)^2} = 0, \\
&r^* = \frac{R_2}{1 + \sqrt{m_2/m_1}} - R_1 \approx 339,260,779 \text{ meters.}
\end{aligned} \tag{2}
$$

The minimal velocity to cause the projectile to reach height r^* can be computed by numerical experiment:

$$
v_0^* \approx 11067.19091 \quad \text{meters per second.} \tag{3}
$$

Derivation

To derive (1), it suffices to write down a competition between the Newton's second law force relation $mr''(t)$ and the sum of two forces due to gravitational attraction for the earth and the moon. Here, m stands for the mass of the projectile.

Gravitational force for the earth

This force, by Newton's universal gravitation law, has magnitude

$$F_1 = \frac{Gm_1 m}{R_3^2}$$

where m_1 is the mass of the earth, G is the universal gravitation constant and R_3 is the distance from the projectile to the center of the earth: $R_3 = R_1 + r(t)$.

Gravitational force for the moon

Similarly, this force has magnitude

$$F_2 = \frac{Gm_2 m}{R_4^2}$$

where m_2 is the mass of the moon and R_4 is the distance from the projectile to the center of the moon: $R_4 = R_2 - R_1 - r(t)$.

Competition between forces

The force equation is

$$\boldsymbol{m r''(t) = -F_1 + F_2}$$

due to the directions of the force vectors. Simplifying the relations and canceling $\boldsymbol{m}$ gives the equation

$$\boldsymbol{r''(t) = -\frac{G m_1}{(R_1 + r(t))^2} + \frac{G m_2}{(R_2 - R_1 - r(t))^2}.} \tag{4}$$

The initial conditions $\boldsymbol{r(0) = 0}$, $\boldsymbol{r'(0) = v_0}$ are obtained from the definition of $\boldsymbol{r(t)}$, which is the distance from the center of mass of the projectile to the earth's surface.

Escape Velocity of the Earth. Let $R = R_1$ below. Gravitational effects F_2 of the moon ignored, the equation for the distance $y(t)$ from the earth's surface to the center of mass of the projectile is given by

$$y''(t) = -\frac{gR^2}{(y(t)+R)^2}, \quad y(0) = 0, \quad y'(0) = v_0. \tag{5}$$

The projectile **escapes** the planet if $y(t) \to \infty$ as $t \to \infty$. The **escape velocity problem** asks which minimal value of v_0 causes escape.
To solve the escape velocity problem, multiply equation (5) by $y'(t)$, then integrate over $[0, t]$ and use the initial conditions $y(0) = 0$, $y'(0) = v_0$ to obtain

$$\frac{1}{2}\left((y'(t))^2 - (v_0)^2\right) = \frac{gR^2}{y(t)+R} - Rg.$$

The square term $(y'(t))^2$ being nonnegative gives the inequality

$$0 \le (v_0)^2 + \frac{2gR^2}{y(t)+R} - 2Rg.$$

If $y(t) \to \infty$, then $v_0^2 \ge 2Rg$. The *escape velocity* minimizes v_0, which occurs at $v_0^2 = 2Rg$, giving

$$v_0 = \sqrt{2gR}. \tag{6}$$

For the earth, $v_0 \approx 11,174$ meters per second, which is slightly more than $25,000$ miles per hour.