Stability of Dynamical systems

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Stability

Consider an autonomous system $\vec{\mathbf{u}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{u}}(t))$ with $\vec{\mathbf{f}}$ continuously differentiable in a region D in the plane.

Stable equilibrium. An equilibrium point $\vec{\mathbf{u}}_0$ in D is said to be **stable** provided for each $\epsilon > 0$ there corresponds $\delta > 0$ such that **(a)** and **(b)** hold:

(a) Given $\vec{\mathbf{u}}(0)$ in D with $\|\vec{\mathbf{u}}(0) - \vec{\mathbf{u}}_0\| < \delta$, then $\vec{\mathbf{u}}(t)$ exists on $0 \le t < \infty$.

(b) Inequality $\|\vec{\mathbf{u}}(t) - \vec{\mathbf{u}}_0\| < \epsilon$ holds for $0 \le t < \infty$.

Unstable equilibrium. The equilibrium point $\vec{\mathbf{u}}_0$ is called **unstable** provided it is **not stable**, which means **(a)** or **(b)** fails (or both).

Asymptotically stable equilibrium. The equilibrium point $\vec{\mathbf{u}}_0$ is said to be **asymptotically stable** provided **(a)** and **(b)** hold (it is **stable**), and additionally

(c) $\lim_{t\to\infty} \|\vec{\mathbf{u}}(t) - \vec{\mathbf{u}}_0\| = 0$ for $\|\vec{\mathbf{u}}(0) - \vec{\mathbf{u}}_0\| < \delta$.

Isolated equilibria

An autonomous system is said to have an **isolated equilibrium** at $\vec{\mathbf{u}} = \vec{\mathbf{u}}_0$ provided $\vec{\mathbf{u}}_0$ is the only constant solution of the system in $|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0| < r$, for $r > 0$ sufficiently small.

Theorem 1 (Isolated Equilibrium)
The following are equivalent for a constant planar system $\vec{\mathbf{u}}'(t) = A\vec{\mathbf{u}}(t)$:

1. The system has an isolated equilibrium at $\vec{\mathbf{u}} = \vec{0}$.
2. $\det(A) \neq 0$.
3. The roots λ_1, λ_2 of $\det(A - \lambda I) = 0$ satisfy $\lambda_1\lambda_2 \neq 0$.

Proof: The expansion $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$ shows that $\det(A) = \lambda_1\lambda_2$. Hence $\mathbf{2} \equiv \mathbf{3}$. We prove now $\mathbf{1} \equiv \mathbf{2}$. If $\det(A) = 0$, then $A\vec{\mathbf{u}} = \vec{0}$ has infinitely many solutions $\vec{\mathbf{u}}$ on a line through $\vec{0}$, therefore $\vec{\mathbf{u}} = \vec{0}$ is not an isolated equilibrium. If $\det(A) \neq 0$, then $A\vec{\mathbf{u}} = \vec{0}$ has exactly one solution $\vec{\mathbf{u}} = \vec{0}$, so the system has an isolated equilibrium at $\vec{\mathbf{u}} = \vec{0}$.

Classification of Isolated Equilibria

For linear equations

$$\vec{\mathbf{u}}'(t) = A\vec{\mathbf{u}}(t),$$

we explain the phase portrait classifications

saddle, node, spiral, center

near an isolated equilibrium point $\vec{\mathbf{u}} = \vec{\mathbf{0}}$, and how to detect these classifications, when they occur.

Symbols λ_1, λ_2 are the roots of $\det(A - \lambda I) = 0$.

Atoms corresponding to roots λ_1, λ_2 happen to classify the phase portrait as well as its stability. A **shortcut** will be explained to determine a classification, *based only on the atoms*.

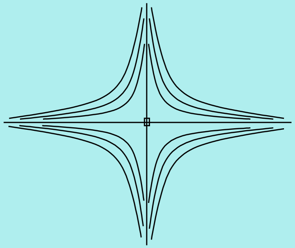

Figure 1. Saddle

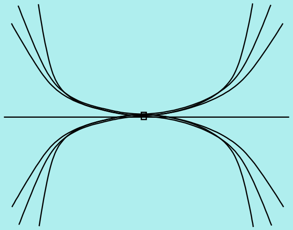

Figure 2. Improper node

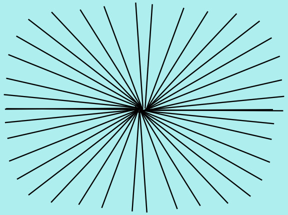

Figure 3. Proper node

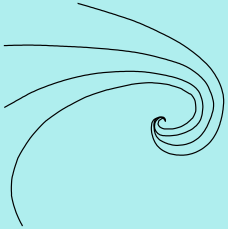

Figure 4. Spiral

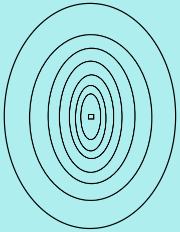

Figure 5. Center

Saddle $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$ real, $\boldsymbol{\lambda}_1 \boldsymbol{\lambda}_2 < 0$

A **saddle** has solution formula

$$\vec{\mathbf{u}}(t) = e^{\lambda_1 t}\vec{\mathbf{c}}_1 + e^{\lambda_2 t}\vec{\mathbf{c}}_2,$$

$$\vec{\mathbf{c}}_1 = \frac{A - \boldsymbol{\lambda}_2 I}{\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2}\vec{\mathbf{u}}(0), \quad \vec{\mathbf{c}}_2 = \frac{A - \boldsymbol{\lambda}_1 I}{\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1}\vec{\mathbf{u}}(0).$$

The phase portrait shows two lines through the origin which are tangents at $t = \pm\infty$ for all orbits.

A saddle is **unstable** at $t = \infty$ and $t = -\infty$, due to the limits of the atoms $e^{r_1 t}$, $e^{r_2 t}$ at $t = \pm\infty$.

Node λ_1, λ_2 real, $\lambda_1\lambda_2 > 0$

Case $\lambda_1 = \lambda_2$. An **improper node** has solution formula

$$\vec{u}(t) = e^{\lambda_1 t}\,\vec{c}_1 + te^{\lambda_1 t}\,\vec{c}_2,$$
$$\vec{c}_1 = \vec{u}(0), \quad \vec{c}_2 = (A - \lambda_1 I)\vec{u}(0).$$

An improper node is further classified as a **degenerate node** ($\vec{c}_2 \neq \vec{0}$) or a **star node** ($\vec{c}_2 = \vec{0}$). Discussed below is **subcase $\lambda_1 = \lambda_2 < 0$**. For **subcase $\lambda_1 = \lambda_2 > 0$**, replace ∞ by $-\infty$.

degenerate node	A phase portrait has all trajectories tangent at $t = \infty$ to direction $\vec{c}_2$.
star node	A phase portrait consists of trajectories $\vec{u}(t) = e^{\lambda_1 t}\vec{c}_1$, a straight line, with limit $\vec{0}$ at $t = \infty$. Vector $\vec{c}_1$ can be any direction.

Node λ_1, λ_2 real, $\lambda_1\lambda_2 > 0$

Case $\lambda_1 \neq \lambda_2$. A **proper node** is any node that is not improper. Its solution formula is

$$\vec{\mathbf{u}}(t) = e^{\lambda_1 t}\vec{\mathbf{c}}_1 + e^{\lambda_2 t}\vec{\mathbf{c}}_2,$$

$$\vec{\mathbf{c}}_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}\vec{\mathbf{u}}(0), \quad \vec{\mathbf{c}}_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}\vec{\mathbf{u}}(0).$$

- A trajectory near a proper node satisfies, for some direction $\vec{\mathbf{v}}$, $\lim_{t\to\omega}\vec{\mathbf{u}}'(t)/|\vec{\mathbf{u}}'(t)| = \vec{\mathbf{v}}$, for either $\omega = \infty$ or $\omega = -\infty$. Briefly, $\vec{\mathbf{u}}(t)$ is tangent to $\vec{\mathbf{v}}$ at $t = \omega$.
- To each direction $\vec{\mathbf{v}}$ corresponds some $\vec{\mathbf{u}}(t)$ tangent to $\vec{\mathbf{v}}$.

Spiral $\lambda_1 = \overline{\lambda}_2 = a + ib$ complex, $a \neq 0$, $b > 0$.

A **spiral** has solution formula

$$\vec{\mathbf{u}}(t) = e^{at}\cos(bt)\,\vec{\mathbf{c}}_1 + e^{at}\sin(bt)\,\vec{\mathbf{c}}_2,$$

$$\vec{\mathbf{c}}_1 = \vec{\mathbf{u}}(0), \quad \vec{\mathbf{c}}_2 = \frac{A - aI}{b}\,\vec{\mathbf{u}}(0).$$

All solutions are bounded harmonic oscillations of natural frequency b times an exponential amplitude which grows if $a > 0$ and decays if $a < 0$. An orbit in the phase plane **spirals out** if $a > 0$ and **spirals in** if $a < 0$.

Center $\lambda_1 = \overline{\lambda}_2 = a + ib$ complex, $a = 0$, $b > 0$

A **center** has solution formula

$$\vec{\mathbf{u}}(t) = \cos(bt)\,\vec{\mathbf{c}}_1 + \sin(bt)\,\vec{\mathbf{c}}_2,$$

$$\vec{\mathbf{c}}_1 = \vec{\mathbf{u}}(0), \quad \vec{\mathbf{c}}_2 = \frac{1}{b} A\vec{\mathbf{u}}(0).$$

All solutions are bounded harmonic oscillations of natural frequency b. Orbits in the phase plane are periodic closed curves of period $2\pi/b$ which encircle the origin.

Attractor and Repeller

- An equilibrium point is called an **attractor** provided solutions starting nearby limit to the point as $t \to \infty$.
- A **repeller** is an equilibrium point such that solutions starting nearby limit to the point as $t \to -\infty$.
- Terms like **attracting node** and **repelling spiral** are defined analogously.

Almost linear systems

A nonlinear planar autonomous system $\vec{\mathbf{u}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{u}}(t))$ is called **almost linear** at equilibrium point $\vec{\mathbf{u}} = \vec{\mathbf{u}}_0$ if there is a 2×2 matrix A and a vector function $\vec{\mathbf{g}}$ such that

$$\vec{\mathbf{f}}(\vec{\mathbf{u}}) = A(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0) + \vec{\mathbf{g}}(\vec{\mathbf{u}}),$$
$$\lim_{\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_0\|\to 0} \frac{\|\vec{\mathbf{g}}(\vec{\mathbf{u}})\|}{\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\|} = 0.$$

The function $\vec{\mathbf{g}}$ has the same smoothness as $\vec{\mathbf{f}}$.

We investigate the possibility that a local phase diagram at $\vec{\mathbf{u}} = \vec{\mathbf{u}}_0$ for the nonlinear system $\vec{\mathbf{u}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{u}}(t))$ is graphically identical to the one for the linear system $\vec{\mathbf{y}}'(t) = A\vec{\mathbf{y}}(t)$ at $\vec{\mathbf{y}} = \mathbf{0}$.

Jacobian Matrix

Almost linear system results will apply to **all isolated equilibria** of $\vec{\mathbf{u}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{u}}(t))$. This is accomplished by expanding f in a Taylor series about each equilibrium point, which implies that the ideas are applicable to different choices of A and g, depending upon which equilibrium point $\vec{\mathbf{u}}_0$ was considered.

Define the **Jacobian matrix** of $\vec{\mathbf{f}}$ at equilibrium point $\vec{\mathbf{u}}_0$ by the formula

$$J = \mathbf{aug}\left(\partial_1 \vec{\mathbf{f}}(\vec{\mathbf{u}}_0), \partial_2 \vec{\mathbf{f}}(\vec{\mathbf{u}}_0)\right).$$

Taylor's theorem for functions of two variables says that

$$\vec{\mathbf{f}}(\vec{\mathbf{u}}) = J(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0) + \vec{\mathbf{g}}(\vec{\mathbf{u}})$$

where $\vec{\mathbf{g}}(\vec{\mathbf{u}})/\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\| \to \mathbf{0}$ as $\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\| \to \mathbf{0}$. Therefore, for $\vec{\mathbf{f}}$ continuously differentiable, we may always take $A = J$ to obtain from the almost linear system $\vec{\mathbf{u}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{u}}(t))$ its **linearization** $y'(t) = A\vec{\mathbf{y}}(t)$.