P40 #1. Solve $\frac{dy}{dx} + 2xy = 0$. Find the implicit and explicit solutions.

$\frac{dy}{y} = -2x\,dx$	Separated form
$\int \frac{dy}{y} = \int -2x\,dx$	method of quadrature
$\ln\lvert y\rvert = -x^2 + c$	Integral tables give the implicit solution.
$\lvert y\rvert = e^{-x^2+c}$	Def of logarithm
$\lvert y\rvert = e^{-x^2}e^{c}$	Rule $e^{a+b} = e^a e^b$
$\pm y = e^c e^{-x^2}$	replace $\lvert y\rvert$ by $\pm y$
$y = (\pm e^c)e^{-x^2}$	move $\pm$ to right side
$y = y_0 e^{-x^2}$	Relabel $\pm e^c$ as y_0.
	Explicit solution found.

Check:

$LHS = y'$	Left side of $\frac{dy}{dx} = -2xy$
$= (y_0 e^{-x^2})'$	substitute explicit sol.
$= y_0 e^{-x^2}(-2x)$	Use $(e^u)' = e^u u'$
$= y(-2x)$	
$= -2xy$	
$= RHS$	DE verified.

P40 #5. Solve $2\sqrt{x}\,\frac{dy}{dx} = \sqrt{1-y^2}$ for implicit, explicit and equilibrium solutions.

Equilibrium Solutions.

$0 = $ RHS of the DE $= \sqrt{1-y^2}$	An equil. sol. is a constant solution.
$0 = 1-y^2$	
$0 = (1-y)(1+y)$	
$y = 1$ or $y = -1$	Equil sols found.

Implicit non-equil sol.

$\frac{y'}{\sqrt{1-y^2}} = \frac{1}{2\sqrt{x}}$	Divide to find separated form.
$\int (1-y^2)^{-\frac{1}{2}} y'\,dx = \frac{1}{2}\int x^{-\frac{1}{2}}\,dx$	method of quadrature
$\sin^{-1} y(x) = x^{\frac{1}{2}} + c$	See #16 integral tables, textbook. Implicit sol found.

Explicit sol.

$y(x) = \sin(x^{\frac{1}{2}} + c)$	Apply sine to both sides of implicit sol above

Note: $y=1$ and $y=-1$ are not included in this formula.

Check: (checks with book, but explain this:)

$LHS = 2x^{\frac{1}{2}}y'$
$= 2x^{\frac{1}{2}}\cos(x^{\frac{1}{2}}+c)\left(\tfrac{1}{2}x^{-\frac{1}{2}}\right)$
$= \cos(x^{\frac{1}{2}}+c)$

$RHS = \sqrt{1-y^2}$
$= \sqrt{1-\sin^2(x^{\frac{1}{2}}+c)}$
$= \pm\cos(x^{\frac{1}{2}}+c)$

Remarks on P40, #11, 17, 19, 27

#11. $y' = xy^3$

The separated form $y'y^{-3} = x$ is easy enough to solve by quadrature, but be aware that $y=0$ is an equil sol, maybe not present in the derived formula.

#17. $y' = 1 + x + y + xy$
$= (1+x)(1+y)$

The separated form $\frac{y'}{1+y} = 1+x$ assumes $y \neq -1$. But $y=-1$ is an equil sol.

#19. $\frac{dy}{dx} = y e^x$

The equil sol $y=0$ can be included in the explicit solution formula, obtained by quadrature on the separated form $\frac{y'}{y} = e^x$.

#27. $\frac{dy}{dx} = 6e^{2x-y}$

There are no equil. solutions. The separated form $e^y y' = 6e^{2x}$ is found by mult of the DE by e^y.

#27 P40 Find equilibria, implicit and explicit solutions for $y' = 6e^{2x-y}$, $y(0)=0$.

Equilibria

Solve $6e^{2x-y} = 0$; no solutions $y = $ constant, so no equilibria.

Implicit Solutions

$e^y y' = 6e^{2x}$	Separated form
$e^y = 3e^{2x} + c$	Integrate
$e^0 = 3e^0 + c$	Set $x=0, y=0$ ($y(0)=$
$\boxed{e^y = 3e^{2x} - 2}$	Substitute back $C=$

Implicit sol.

Explicit Solution

Solve for y as a function of x in the implicit form

$e^y = 3e^{2x} - 2$	Given above
$y = \ln\lvert 3e^{2x} - 2\rvert$	Take logs

Answer Check

$e^y = 3e^{2x} - 2$
$e^y y' = 3(2)e^{2x}$ Diff both sides
$y' = 6e^{2x}/e^y$ Divide
$= 6e^{2x-y}$

$y(0) = \ln\lvert 3e^0 - 2\rvert$
$= \ln 1$
$= 0$
IC checks also.

/ du

Solve the linear problem

$$y' = y + e^x$$

std. form	
$y' + (-1)y = e^x$	Form $y' + p(x)y = g(x)$
Factor $e^{\mathbb{P}}$	
$\mathbb{P} = \int p(x)dx$	
$= \int (-1)dx$	
$= -x$	Drop constant of integration
$e^{\mathbb{P}} = e^{-x}$	simplified integrating factor
Quadrature form	
$y' + (-1)y = e^x$	std. form from above.
$\dfrac{(e^{\mathbb{P}}y)'}{e^{\mathbb{P}}} = e^x$	Replace LHS by $\dfrac{(e^{\mathbb{P}}y)'}{e^{\mathbb{P}}}$.
$\dfrac{(e^{-x}y)'}{e^{-x}} = e^x$	Replace $e^{\mathbb{P}}$ by e^{-x}.
$(e^{-x}y)' = 1$	Cross-multiply, simplify to quadrature form.
method of quadrature	
$\int (e^{-x}y)'dx = \int dx$	Integrate both sides of the quadrature form.
$e^{-x}y = x + c$	
$\boxed{y = ce^x + xe^x}$	General solution $y = y_h + y_p$.

...will check next...

P53 # 3. $y' + 3y = 2xe^{-3x}$. Solve by the factorization method.

$y' + (3)y = 2xe^{-3x}$	standard form $y' + py = g$
$\mathbb{P} = \int (3)dx$	primitive $\mathbb{P} = \int p\,dx$
$= 3x$	simplify constants
$e^{\mathbb{P}} = e^{3x}$	simplified $e^{\mathbb{P}}$

Find Quadrature form

$y' + (3)y = 2xe^{-3x}$	std form
$\dfrac{(e^{\mathbb{P}}y)'}{e^{\mathbb{P}}} = 2xe^{-3x}$	Replace $y' + py$ by $\dfrac{(e^{\mathbb{P}}y)'}{e^{\mathbb{P}}}$
$\dfrac{(e^{3x}y)'}{e^{3x}} = 2xe^{-3x}$	Substitute $e^{\mathbb{P}} = e^{3x}$
$(e^{3x}y)' = 2x$	use $e^{3x}e^{-3x} = e^0 = 1$ after cross-multiply

Apply method of quadrature

$\int (e^{3x}y)'dx = \int 2x\,dx$	Apply quadrature to the quadrature form above
$e^{3x}y = x^2 + c$	Fund. Thm. of calculus
$y = (x^2+c)e^{-3x}$	Divide

Report ans and check

$$\boxed{y = (x^2+c)e^{-3x}}$$

ans checks with textbook

P 53 #5 $xy' + 2y = 3x$, $y(1) = 5$ Solve by the factorization method.

$y' + (\frac{2}{x})y = 3$ Standard form $y' + py = q$

$\mathbb{P} = \int (\frac{2}{x})dx$ primitive $\mathbb{P} = \int p\,dx$

$= 2\ln x$

$= \ln x^2$

$e^{\mathbb{P}} = e^{\ln x^2}$

$= x^2$ Simplified $e^{\mathbb{P}}$ found

Find Quadrature Form

$y' + (\frac{2}{x})y = 3$ Std form copied

$\frac{(e^{\mathbb{P}}y)'}{e^{\mathbb{P}}} = 3$ Replace $y' + py$ by $\frac{(e^{\mathbb{P}}y)'}{e^{\mathbb{P}}}$

$\frac{(x^2y)'}{x^2} = 3$ Substitute x^2 for $e^{\mathbb{P}}$

$(x^2y)' = 3x^2$ Quadrature form found

Method of Quadrature

$\int (x^2y)'dx = \int 3x^2dx$ Apply quadrature to the previous line.

$x^2y = x^3 + c$

$y = x + c/x^2$ Divide. Solution Candidate found.

Report answer and Check

$5 = 1 + \frac{c}{1^2}$ Substitute $x=1, y=5$ to find $c=4$.

answer $\boxed{y = x + \frac{4}{x^2}}$

answer checks with text.

Existence – Uniqueness

P22 **Theorem.** The problem

$$\begin{cases} y' = f(x,y) \\ y(x_0) = x_0 \end{cases}$$

has one and only one solution $y(x)$ defined in an interval $|x - x_0| < h$, provided $h > 0$ is sufficiently small, f and f_y are continuous.

Engineering impact: Models $y' = f(x,y)$ with continuous f, f_y have answers that can be computed with DE solver software. Sometimes, computer algebra systems can find an explicit answer.

Warning. Simple models like

$$\begin{cases} y' = 3y^{2/3} \\ y(0) = 0 \end{cases}$$

This problem has ∞-many solutions.

can be fed into computer algebra systems and numeric software packages. This problem has no "answer" but in all cases the computer gives an answer and no complaint.

The lesson is that computers are stupid, and engineers have to supply the logic and intuition, in order to make use of them.