Differential Equations and Linear Algebra 2250

Exam 3 Practice Problems

28 Nov 2010, Part II: Problems 3,4,5

3. (Chapter 5)

(3a) [70%] Find the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 2x = 5\sin(t)$.

Answer:

$x = \sin t - 2\cos t$

(3b) [70%] Write the general solution of the forced spring-mass system $x'' + 4x = 5\sin(t)$ as the sum of two harmonic oscillations of different frequencies.

Answer:

$x_p = \frac{5}{3}\sin t$, $x_h = c_1\cos 2t + c_2\sin 2t$, $x = x_h + x_p$. The frequencies are 1 and 2.

(3c) [70%] Determine without solving the unforced mass-dashpot-spring mechanical system $4x''(t)+10x'(t)+3x(t) = 0$ it's classification of over-damped, critically damped, or under-damped.

Answer:

The discriminant for $4r^2 + 10r + 3 = 0$ is $D = 10^2 - 4(4)(3) = 52$. It's over-damped.

(3d) [40%] Find by variation of parameters an integral formula for a particular solution x_p of the equation $x'' + 4x' + 20x = e^{t^2}\ln(t^2 + 1)$. To save time, don't try to evaluate integrals (it's impossible).

Answer:

$x_p(t) = x_1(t)\int_0^t k1(u)f(u)du + x_2(t)\int_0^t k2(u)f(u)du$, $f(t) = e^{t^2}\ln(t^2+1)$, $x_1(u) = e^{-2u}\cos(4u)$, $x_2(u) = \frac{1}{2}e^{-2u}\sin(4u)$, $W(u) = 2e^{-4u}$, $k_1(u) = -x_2(u)/W(u) = -0.5e^{2u}\sin(4u)$, $k_2(u) = x_1(u)/W(u) = 0.5e^{2u}\cos(4u)$. On paper is expected the equation

$$y_p = e^{-2t}\cos(4t)\left(\int_0^t -\frac{1}{2}e^{2u}\sin(4u)e^{u^2}\ln(u^2+1)du\right) + e^{-2t}\sin(4t)\left(\int_0^t \frac{1}{2}e^{2u}\cos(4u)e^{u^2}\ln(u^2+1)du\right)$$

(3e) [30%] Determine the practical resonance frequency ω for the electric current equation

$$2I'' + 7I' + 50I = 100\omega\cos(\omega t).$$

Answer:

$\omega = 1/\sqrt{LC} = 1/\sqrt{2/50} = \sqrt{25} = 5.$

(3f) [30%] Determine the practical resonance frequency ω for the mass-dashpot-spring mechanical system

$$2x''(t) + 7x'(t) + 50x(t) = 100\cos(\omega t).$$

Answer:

$\omega = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}} = \sqrt{25 - \frac{49}{8}} = \frac{1}{4}\sqrt{302}.$

(3g) [30%] Find the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 10x = 5\cos(t)$.

Answer:

$x(t) = (2/17)\sin(t) + (9/17)\cos(t)$ by undetermined coefficients or Laplace theory.

4. **(Chapter 5)**

(4a) [70%] Find a particular solution $y_p(x)$ and the homogeneous solution $y_h(x)$ for $\dfrac{d^4y}{dx^4} - \dfrac{d^2y}{dx^2} = 12x^2$.

Answer:

Use undetermined coefficients to solve for $y_p = -x^4 - 12x^2$. The homogeneous solution is $y_h = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$.

(4b) [30%] Find the shortest trial solution in the method of undetermined coefficients for the differential equation $y'' + y = 3\cos x$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

Answer:

The trial solution is a linear combination of the atoms $x\cos x, x\sin x$. Undetermined coefficient rules were applied to the atoms of $f(x) = 3\cos x$, divided into Group 1: $\cos x$ and Group 2: $\sin x$. There was a conflict with the atoms $\cos x, \sin x$ of the homogeneous equation, resolved by multiplication of each of the two groups by x.

(4c) [50%] Assume $f(x)$ is a linear combination of atoms computed by Euler's theorem from the characteristic equation is $r^2(r+1)(r^2+9) = 0$. Find the shortest trial solution in the method of undetermined coefficients for the differential equation $y''' - y' = f(x)$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

Answer:

$f(x)$ is a linear combination of atoms divided into Group 1: $1, x$, Group 2: e^{-x}, Group 3: $\cos 3x$, Group 4: $\sin 3x$. The homogeneous equation $y''' - y' = 0$ has atoms 1, e^x, e^{-x}. The trial solution is a linear combination of the atoms x, x^2, xe^{-x}, $\cos 3x, \sin 3x$. Undetermined coefficient rules were used to resolve the conflicts in Group 1 and Group 2. The other two groups were unchanged.

(4d) [50%] Find the shortest trial solution in the method of undetermined coefficients for the differential equation $y''' - y' = x + e^x$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

Answer:

Characteristic equation $r^3 - r = 0$ has roots $0, 1, -1$. Function $f(x) = x + e^x$ has 3 atoms divided into Group 1: $1, x$, Group 2: e^x. The trial solution is a linear combination of the atoms x, x^2, xe^x. Undetermined coefficient rules were used to resolve the conflicts in Group 1 and Group 2.

(4e) [25%] The general solution of a certain linear homogeneous differential equation with constant coefficients is

$$y = c_1 e^{-2x} + c_2 x e^{-2x} + c_3 + c_4 x + c_5 x^2 + c_6 e^x.$$

Find the factored form of the characteristic polynomial.

Answer:

The atoms are constructed from roots $-2, -2, 0, 0, 0, 1$, listed according to multiplicity. Then $(r+2)^2$, r^3 and $(r-1)$ are factors. The characteristic polynomial is $a(r+2)^2 r^3 (r-1)$ for some nonzero constant a.

(4f) [30%] Find five independent solutions of a homogeneous linear constant coefficient differential equation whose sixth order characteristic equation has roots 1, 1, 1, 0, $1+i$, $1-i$.

Answer:

According to Euler's theorem, a basis is the list of atoms $e^x, xe^x, x^2e^x, e^x\cos x, e^x \sin x$. Choose five of them. Cite a theorem: *Distinct atoms are independent.*

(4g) [25%] Let $f(x) = 4x^5e^x$. Find a constant-coefficient linear homogeneous differential equation of smallest order which has $f(x)$ as a solution.

Answer:

The atom x^5e^x is constructed from roots $1,1,1,1,1,1$, listed according to multiplicity. Then the characteristic polynomial must include factor $(r-1)^6$. The characteristic polynomial must be a constant multiple of $(r-1)^6 = r^6 - 6r^5 + 15r^4 - 20r^3 + 15r^2 - 6r + 1$. This characteristic equation belongs to the differential equation $y^{(6)} - 6y^{(5)} + 15y^{(4)} - 20y^{(3)} + 15y'' - 6y' + y = 0$.

5. **(Chapter 6)**

(5a) [80%] Define $A = \begin{pmatrix} 4 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Find $A^3\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ without using matrix multiply.

Answer:

First find the eigenpairs of A, $(\lambda_i, \mathbf{v}_i)$. They are $\lambda_1 = \lambda_2 = 4$, $\lambda_3 = 2$,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Vector $\mathbf{v} \equiv \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \mathbf{v}_2 + \mathbf{v}_3$, where $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are the eigenvectors in Fourier's model

$$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3.$$

Then $A^3\mathbf{v} = A((0)\mathbf{v}_1 + (1)\mathbf{v}_2 + (1)\mathbf{v}_3) = c_1\lambda_1^3\mathbf{v}_1 + c_2\lambda_2^3\mathbf{v}_2 + c_3\lambda_3^3\mathbf{v}_3 = 4^3\mathbf{v}_2 + 2^3\mathbf{v}_3 = \begin{pmatrix} 16 \\ 56 \\ 72 \end{pmatrix}$.

(5b) [40%] Given $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, which has eigenvalues $1, 1, -1$, assume there exists an invertible matrix P and a diagonal matrix D such that $AP = PD$. Circle those vectors from the list below which are possible columns of P.

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Answer:

Matrix P must contain eigenvectors of P corresponding to eigenvalues 1, 1, -1, in some order. For each given vector $\mathbf{v}$, multiply $A\mathbf{v}$ and see if it is $\lambda\mathbf{v}$ for some λ. The first fails. The second works for $\lambda = 1$. The third fails.

(5c) [40%] Find all eigenpairs for the matrix $A = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}$. Display the matrices P, D in the diagonalization equation $AP = PD$. Finally, display Fourier's model.

Answer:

Eigenpairs are $\left(-1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$, $\left(1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$. The matrices P, D are defined by

$$P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Fourier's model is

$$A\left(c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = c_1(-1)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2(1)\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(5d) [50%] Find the remaining eigenpairs of

$$E = \begin{pmatrix} 6 & 2 & -2 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

provided we already know one eigenpair

$$\left(6, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right).$$

Answer:

Eigenvalues are 4, 6, 6 with corresponding eigenvectors $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

(5e) [40%] Suppose a 3×3 matrix A has three eigenpairs

$$\left(3, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(0, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

Find the nine entries of A from the eigenanalysis equation $AP = PD$.

Answer:

Define $P = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Solution 1. Then $AP = PD$, which implies $A = PDP^{-1}$. Find the inverse of A from the augmented matrix of P and I, $P^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Multiply

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solution 2. Let the eigenpairs be labeled as $(\lambda_i, \mathbf{v}_i)$, $i = 1, 2, 3$. Replace $\mathbf{v}_1$, $\mathbf{v}_2$ by eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, which are linear combinations of the given eigenvectors. Then $AP = PD$ becomes $AI = ID = D$, so $A = D$.

(5f) [25%] Assume the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = A\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Display Fourier's model for the 3×3 matrix A.

Answer:

$$A \left(c_1 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = 0c_1 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + 2c_2 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + 2c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(5g) [30%] Find the eigenvalues of the matrix $A = \begin{pmatrix} -2 & 7 & 1 & 27 \\ -1 & 6 & -3 & 62 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$. To save time, **do not** find eigenvectors!

Answer:

Expand by cofactors along column 1. The eigenvalues are $-1, 1, 2, 5$.

(5h) [30%] Assume A is 2×2 and Fourier's model holds:

$$A \left(c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = 2c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find A.

Answer:

$AP = PD$ implies $A = PDP^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} .5 & .5 \\ .5 & -.5 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

(5i) [40%] Let $A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Circle the possible eigenvectors of A in the list below.

$$\begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Answer:

Fourier's model does not hold [A is not diagonalizable] because there are only two eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ for eigenvalue $\lambda = 3$. The first is a linear combination of these eigenvectors, hence itself an eigenvector. The second is one already reported, The third is not an eigenvector. The problem should be solved by testing the equation $A\mathbf{v} = 3\mathbf{v}$ for each of the 3 vectors v in the list, not by doing the eigenanalysis of A.

(5j) [40%] Consider the 3×3 matrix

$$E = \begin{pmatrix} 4 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Show that matrix E has a Fourier model: [original had a typo]

$$E\left(c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\end{pmatrix} + c_3\begin{pmatrix}2\\-1\\1\end{pmatrix}\right) = 4c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + 4c_2\begin{pmatrix}0\\1\\1\end{pmatrix} + 2c_3\begin{pmatrix}2\\-1\\1\end{pmatrix}.$$

Answer:

Do the eigenanalysis of A. Alternate: verify that the eigenpairs extracted from Fourier's model actually work, which involves 3 matrix multiplies.

(5k) [20%] Let $P = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$ and define A by $AP = PD$. Display the eigenpairs of A.

Answer:

$\left(3, \begin{pmatrix}3\\1\end{pmatrix}\right)$, $\left(-2, \begin{pmatrix}1\\-1\end{pmatrix}\right)$

(5m) [20%] Assume the vector general solution $\vec{\mathbf{u}}(t)$ of the 2×2 linear differential system $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$ is given by

$$\vec{\mathbf{u}}(t) = c_1 e^{2t}\begin{pmatrix}1\\-1\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}2\\1\end{pmatrix}.$$

Find the matrix C.

Answer:

The eigenvalues come from the exponents in the exponentials, 2 and 2. The eigenpairs are $\left(2, \begin{pmatrix}1\\-1\end{pmatrix}\right)$, $\left(2, \begin{pmatrix}2\\1\end{pmatrix}\right)$. Then $P = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Solve $CP = PD$ to find $C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. The usual eigenpairs for C are the columns of the identity. But the eigenvalues are equal, therefore any linear combination of the two eigenvectors is also an eigenvector. This justifies the correctness of the strange eigenpairs given in the problem.