Name ____ **Lecture Meets at** ______

Differential Equations and Linear Algebra 2250

Midterm Exam 3 [12:25 lecture]
Version 2.12.2010

Scores
3.
4.
5.

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

3. **(Chapter 5)** Complete all.

(3a) [40%] Write the solution $x(t)$ of

$$x''(t) + 9x(t) = 21\sin(4t), \quad x(0) = x'(0) = 0,$$

as the sum of two harmonic oscillations of different natural frequencies.
Report the two natural frequencies and the solution $x(t)$.
To save time, don't convert to phase-amplitude form.

Answer:
The frequencies are 4 and 3, the solution is $x(t) = -3\sin(4t) + 4\sin(3t)$.

(3b) [30%] Given $14x''(t) + 41x'(t) + 15x(t) = 0$, which represents a damped spring-mass system with $m = 14$, $c = 41$, $k = 15$, determine if the equation is over-damped , critically damped or under-damped.
To save time, do not solve for $x(t)$!

Answer:
Use the quadratic formula to decide. The number under the radical sign in the formula, called the discriminant, is $b^2 - 4ac = 41^2 - 4(14)(15) = 841$, therefore there are two distinct roots and the equation is **over-damped**. Alternatively, factor $14r^2 + 41r + 15$ to obtain roots $-3/7$, $-5/2$ and then classify as **over-damped**.

(3c) [30%] Consider the variation of parameters formula (33) in Edwards-Penney,

$$y_p(x) = y_1(x)\left(\int \frac{-y_2(x)f(x)}{W(x)}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(x)}dx\right).$$

Given the second order equation

$$y''(x) + 34y'(x) + 298y(x) = 7\cos(x^2) + 13\sin(x),$$

write the equations for the variables y_1, y_2, W, f.
To save time, do not write out y_p and do not try to evaluate any integrals.

Answer:
Variables are $y_1(x) = e^{-17x}\cos(3x)$, $y_2(x) = e^{-17x}\sin(3x)$, $f(x) = 7\cos(x^2) + 13\sin(x)$, $W(x) = 3e^{-34x}$.

(3c) [30%] (Alternate version announced at exam time)

$$y''(x) + 34y'(x) + 289y(x) = 7\cos(x^2) + 13\sin(x),$$

Answer:
Variables are $y_1(x) = e^{-17x}$, $y_2(x) = xe^{-17x}$, $f(x) = 7\cos(x^2) + 13\sin(x)$, $W(x) = e^{-34x}$.

Use this page to start your solution. Attach extra pages as needed.

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4. **(Chapter 5)** Complete all.

(4a) [60%] A homogeneous linear differential equation with constant coefficients has characteristic equation of order 9 with roots $0, 0,\ 3, 3, 3,\ 2i, -2i,\ 2i,\ -2i$, listed according to multiplicity. The corresponding non-homogeneous equation for unknown $y(x)$ has right side $f(x) = 4e^x + 5e^{3x} + 6x^3 + 7x\sin 2x$. Determine the undetermined coefficients **shortest** trial solution for y_p.
To save time, do not evaluate the undetermined coefficients and do not find $y_p(x)$! Undocumented detail or guessing earns no credit.

Answer:

The atom list for $f(x)$ is e^x, e^{3x}, $1, x, x^2, x^3$, $\cos 2x$, $x\cos 2x$, $\sin 2x$, $x\sin 2x$. The list of 10 atoms is broken into 5 groups, each group having exactly one base atom: (1) e^x, (2) e^{3x}, (3) $1, x, x^2, x^3$, (4) $\cos 2x$, $x\cos 2x$ (5) $\sin 2x$, $x\sin 2x$. The modification rule is applied to groups 3 through 5. The trial solution is a linear combination of the replacement 10 atoms in the new list (1) e^x, (2) x^3e^{3x}, (3) x^2, x^3, x^4, x^5, (4) $x^2\cos 2x$, $x^3\cos 2x$ (5) $x^2\sin 2x$, $x^3\sin 2x$. Groups 1 is not modified.

(4b) [40%] Let $f(x) = 4x^2e^{2x} + xe^{-x}\sin 2x$. Find the roots of the characteristic polynomial of a constant-coefficient linear homogeneous differential equation which has $f(x)$ as a solution.

Answer:

Because $xe^{-x}\sin 2x$ is an atom for the differential equation if and only if $e^{-x}\sin 2x$, $xe^{-x}\sin 2x$, $e^{-x}\cos 2x$ $xe^{-x}\cos 2x$ are atoms. Then the characteristic equation must have roots $-1 + 2i, -1 - 2i, -1 + 2i, -1 - 2i$, listing according to multiplicity. Similarly, x^2e^{2x} is an atom for the differential equation if and only if 2 is a triple root of the characteristic equation. Total of 7 roots: $2, 2, 2, -1 \pm 2i, -1 \pm 2i$ with product of the factors $(r-2)^3((r+1)^2+4)((r+1)^2+4)$ equal to the 7th order characteristic polynomial.

Use this page to start your solution. Attach extra pages as needed.

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5. (Chapter 6) Complete all parts.

(5a) [40%] Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 10 & 20 & 0 & 3 \\ 30 & 40 & -3 & 0 \end{pmatrix}$.

To save time, do not find eigenvectors!

Answer:

$-1, 5, \pm 3i$. The block matrix determinant theorem can be used, or the cofactor method.

(5b) [30%] Given $A = \begin{pmatrix} 8 & 1 & -2 \\ 0 & 9 & 0 \\ -1 & 1 & 7 \end{pmatrix}$, which has eigenvalues $6, 9, 9$, find all eigenvectors for eigenvalue 9.

To save time, do not find the eigenvector for eigenvalue 6.

Answer:

One frame sequence is required for $\lambda = 3$. The sequence starts with $\begin{pmatrix} -1 & 1 & -2 \\ 0 & 0 & 0 \\ -1 & 1 & -2 \end{pmatrix}$, the last frame having two rows of zeros. There are two invented symbols t_1, t_2 in the last frame algorithm answer $x_1 = t_1 - 2t_2$, $x_2 = t_1$, $x_3 = t_2$. Taking ∂_{t_1} and ∂_{t_2} gives two eigenvectors, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

(5c) [40%] Find the matrices P, D in the diagonalization equation $AP = PD$ for the matrix $A = \begin{pmatrix} 3 & -1 \\ 8 & -3 \end{pmatrix}$.

Answer:

The eigenpairs are $\left(-1, \begin{pmatrix} 1 \\ 4 \end{pmatrix}\right)$, $\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$. Then $P = \begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Use this page to start your solution. Attach extra pages as needed.