Name KEY version 2

Scores
1.
2.
3.

Differential Equations and Linear Algebra 2250

Midterm Exam 1

Version 2, 16 Feb 2012

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**

   (a) [25%] Solve $y' = \dfrac{1-x}{1+x}$.

   (b) [25%] Solve $y' = (\cos x + 2)(\cos x - 2)$.

   (c) [25%] Solve $y' = x^2 \cos\left(x^3\right)$, $y(0) = 1$.

   (d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}\left(e^{3t}v(t)\right) = 12e^{3t}$, $v(0) = 0$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(0) = \frac{304}{3}$.

(a) $y' = \dfrac{1-x}{1+x} = -1 + \dfrac{2}{1+x}$ by long division, $\Rightarrow y = -x + 2\ln|1+x| + c$

(b) $y' = \cos^2 x - 4 = \dfrac{1+\cos 2x}{2} - 4 \Rightarrow y = -\frac{7}{2}x + \frac{1}{4}\sin(2x) + c$

(c) $y' = \frac{1}{3}(3x^2)\cos(x^3) = \frac{1}{3}\cos(u)du$ where $u = x^3$. Then

$y = \frac{1}{3}\sin u + c = \frac{1}{3}\sin(x^3) + c$. Because $y(0) = 1$, Then $c = 1$.

$y = 1 + \frac{1}{3}\sin(x^3)$

(d) By quadrature, $e^{3t}v = 4e^{3t} + c$. Then $v(0) = 0$ implies $c = -4$.

we solved for $v(t) = 4 - 4e^{-3t}$. Now solve

$$\begin{cases} x' = 4 - 4e^{-3t} \\ x(0) = \frac{304}{3} \end{cases}$$

By quadrature,

$x = 4t + \frac{4}{3}e^{-3t} + c$

$\frac{304}{3} = 0 + \frac{4}{3} + c$

$100 = c$

$x(t) = 4t + \frac{4}{3}e^{-3t} + 100$

Use this page to start your solution. Attach extra pages as needed.

Name. KEY Version 2

2. **(Classification of Equations)**
The differential equation $y' = f(x, y)$ is defined to be **separable** provided $f(x, y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check ($\boxed{X}$) the problems that can be converted into separable form. No details expected.

$\boxed{X}$ $y' + xy = y(y + x) - x^2y^2$	$\boxed{}$ $y' = (x - 1)(y + 1) + y^2$
$\boxed{}$ $y' = \cos(x + y)$	$\boxed{}$ $e^x y' = y + x^2$

(b) [10%] State a partial derivative test that decides if $y' = f(x, y)$ is a quadrature differential equation.

(c) [20%] Apply classification tests to show that $y' = x + y^2$ is not a linear differential equation. Supply all details.

(d) [30%] Apply a test to show that $y' = e^y + \tan|x|$ is not separable. Supply all details.

(a) $y' + xy = y^2 + xy - x^2y^2 \Rightarrow y' = y^2 - x^2y^2 = (1-x^2)y^2 = FG$

$y' = (x-1)(y+1) + y^2 \Rightarrow y' = xy - y + x - 1 + y^2$ not separable

$\frac{f_x}{f} = \frac{y+1}{xy - y + x - 1 + y^2}$ Substitute $x = 1$, $\frac{f_x}{f} = \frac{y+1}{y^2}$ depends on y

$y' = \cos(x+y)$ not separable

$\frac{f_x}{f} = \frac{-\sin(x+y)}{\cos(x+y)} = -\tan(x+y)$ depends on y

$e^x y' = y + x^2$ not separable

$\frac{f_y}{f} = \frac{e^{-x}}{(y + x^2)e^{-x}} = \frac{1}{y + x^2}$ depends on x

(b) $y' = f(x,y)$ is quadrature $\Leftrightarrow \frac{\partial f}{\partial y} = 0$

(c) $\frac{f_x}{f} = \frac{1}{x + y^2}$ depends on $y \Rightarrow$ not separable (not requested)

$f_y = 2y$ depends on $y \Rightarrow$ not linear DE

(d) Let $f(x,y) = e^y + \tan|x|$. For $x > 0$, $x = 0$, $x < 0$,

$\frac{f_y}{f} = \frac{e^y}{e^y + \tan|x|}$. Substitute $y = 0$, Then $\frac{f_y}{f} = \frac{1}{1 + \tan|x|}$ depends on x

Use this page to start your solution. Attach extra pages as needed.

$\Rightarrow$ Not separable.

Name. KEY Version 2

3. (Solve a Separable Equation)
Given $(xy + y)y' = ((x+1)\sin(x) + x)(y+2)(y-2)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

Separated form

$y' = FG$

use long division

$F = \sin x + \frac{x}{1+x} = \sin x + 1 + \frac{-1}{1+x}$

$G = \frac{(y+2)(y-2)}{y}$

$\frac{y'}{G} = F$

$\int F\,dx = -\cos x + x - \ln|1+x| + c_1$

$\int \frac{y'}{G}\,dx = \int \frac{yy'\,dx}{(y+2)(y-2)}$

$= \int \left(\frac{A}{y+2} + \frac{B}{y-2}\right) y'\,dx$

$= \frac{1}{2}\ln|y+2| + \frac{1}{2}\ln|y-2| + c_2$

partial fractions

$\frac{y}{(y+2)(y-2)} = \frac{A}{y+2} + \frac{B}{y-2}$

$y = A(y-2) + B(y+2)$

$\begin{cases} B = \frac{1}{2} \\ A = \frac{1}{2} \end{cases}$

$\int \frac{y'\,dx}{G} = \int F\,dx$

$\Rightarrow \frac{1}{2}\ln|y+2| + \frac{1}{2}\ln|y-2| = -\cos x + x - \ln|1+x| + C$

Use this page to start your solution. Attach extra pages as needed.