Name KEY version 1

Differential Equations and Linear Algebra 2250

Midterm Exam 1

Version 1, 16 Feb 2012

Scores
1.
2.
3.

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**

   (a) [25%] Solve $y' = \dfrac{x - x^2}{1+x}$.

   (b) [25%] Solve $y' = (\cos x + 1)(\cos x - 1)$.

   (c) [25%] Solve $y' = x^2 \sin\left(x^3\right)$, $y(0) = 1$.

   (d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}\left(e^{2t}v(t)\right) = 10e^{2t}$, $v(0) = 0$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(0) = \frac{205}{2}$.

(a) By long division, $y' = 2 - x - \frac{2}{1+x} \Rightarrow y = 2x - \frac{1}{2}x^2 - 2\ln|1+x| + c$

(b) $y' = \cos^2 x - 1 = \frac{1+\cos 2x}{2} - 1 \Rightarrow y = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

(c) $y' = \frac{1}{3}(3x^2)\sin(x^3) = \frac{1}{3}\sin(u)\,du$ where $u = x^3$. Then

$y = -\frac{1}{3}\cos(u) + c = -\frac{1}{3}\cos(x^3) + c$. Then $y(0) = 1 \Rightarrow c = 4/3$

$y = -\frac{1}{3}\cos(x^3) + \frac{4}{3}$

(d) $(e^{2t}v)' = 10e^{2t} \Rightarrow$ by quadrature $e^{2t}v = 5e^{2t} + c$. Then

$v(0) = 0 \Rightarrow c = -5$ and $v = 5 - 5e^{-2t}$. Next, Solve

$$\begin{cases} x' = 5 - 5e^{-2t} \\ x(0) = \frac{205}{2} \end{cases}$$

by quadrature

$x = 5t + \frac{5}{2}e^{-2t} + c$, and $x(0) = \frac{205}{2} \Rightarrow \frac{205}{2} = \frac{5}{2} + c$

$\Rightarrow c = 100$. Then

$x(t) = 5t + \frac{5}{2}e^{-2t} + 100$

Use this page to start your solution. Attach extra pages as needed.

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2. (Classification of Equations)
The differential equation $y' = f(x, y)$ is defined to be **separable** provided $f(x, y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check ($\boxed{\text{X}}$) the problems that can be converted into separable form. No details expected.

X	$y' + xy = y(y + x) - xy^2$	X	$y' = (x-1)(y+1) + (1-x)y^2$
	$y' = \cos(x + y)$	X	$e^x y' = xy + x^2 y$

(b) [10%] State a partial derivative test that decides if $y' = f(x, y)$ is a quadrature differential equation.

(c) [20%] Apply classification tests to show that $y' = x + y^2$ is not a linear differential equation. Supply all details.

(d) [30%] Apply a test to show that $y' = e^y + \ln|1 + x|$ is not separable. Supply all details.

(a) $y' + xy = y^2 + xy - xy^2 \Rightarrow y' = (1-x)y^2 = FG$

$y' = (x-1)(y+1) + (1-x)y^2 \Rightarrow y' = (x-1)(y+1-y^2) = FG$

$y' = \cos(x+y)$ not separable: $\dfrac{f_y}{f} = \dfrac{-\sin(x+y)}{\cos(x+y)} = -\tan(x+y)$

$e^x y' = xy + x^2 y \Rightarrow y' = \left(\dfrac{x + x^2}{e^x}\right) y = FG$

(b) $\dfrac{\partial f}{\partial y} = 0 \iff y' = f(x,y)$ is quadrature DE.

(c) $\dfrac{\partial f}{\partial y}$ indep of $y \iff y' = f(x,y)$ linear DE. Here, $f(x,y) = x + y^2$ with $\dfrac{\partial f}{\partial y} = 2y$ depends on $y \Rightarrow$ not linear

(d) Let $f(x,y) = e^y + \ln|1+x|$. Then

$\dfrac{f_y}{f} = \dfrac{e^y}{e^y + \ln|1+x|}$ at $y=0$, $\dfrac{f_y}{f} = \dfrac{1}{1 + \ln|1+x|}$ depends on x

$\Rightarrow y' = f(x,y)$ is not separable.

Use this page to start your solution. Attach extra pages as needed.

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3. **(Solve a Separable Equation)**
Given $(xy+y)y' = \left((x+1)\sin(x)+x^2\right)(y+2)(y-2)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

Separated form

$y' = FG$ $\quad F = \sin x + \dfrac{x^2}{1+x}$

$\quad\quad = \sin x + x - 1 + \dfrac{1}{x+1}$ by long division

$\quad G = \dfrac{(y+2)(y-2)}{y}$

$\dfrac{y'}{G} = F$ $\quad$ Apply quadrature

$$\int F\,dx = -\cos x + \frac{x^2}{2} - x + \ln|1+x| + C_1$$

$$\int \frac{y'}{G}\,dx = \int \frac{y y'\,dx}{(y+2)(y-2)}$$

$$= \int\left(\frac{A}{y+2} + \frac{B}{y-2}\right) y'\,dx$$

$$= A\ln|y+2| + B\ln|y-2| + C_2$$

$$= \frac{1}{2}\left(\ln|y+2| + \ln|y-2|\right) + C_2$$

partial fractions

$$\frac{y}{(y+2)(y-2)} = \frac{A}{y+2} + \frac{B}{y-2}$$

$$y = A(y-2) + B(y+2)$$

$$\begin{cases} B = \frac{1}{2} \\ A = \frac{1}{2} \end{cases}$$

answer

$$\ln|y+2| + \ln|y-2| = -2\cos x + x^2 - 2x + 2\ln|1+x| + c$$

Use this page to start your solution. Attach extra pages as needed.