

Piecewise Representation of Switching Inputs

- Unit step.
- Pulse function.
- Ramp function.
- Piecewise-defined functions.
- Laplace Pulse Example.
- Laplace of a piecewise-defined function.

Unit Step Function

Definition:

$$\mathbf{u}(t - a) = \begin{cases} 1 & t \geq a, \\ 0 & t < a. \end{cases}$$

Example: Write the piecewise defined function $f(t)$ in terms of unit step functions.

$$f(t) = \begin{cases} \sin t & t \geq \pi, \\ 0 & t < \pi, \end{cases}$$

Solution:

$$\begin{aligned} f(t) &= \sin(t) \begin{cases} 1 & t \geq \pi, \\ 0 & t < \pi. \end{cases} \\ &= \sin(t)\mathbf{u}(t - \pi). \end{aligned}$$

Pulse Function

Definition:

$$\begin{aligned}\text{pulse}(t, a, b) &= \begin{cases} 1 & a \leq t < b, \\ 0 & t < a, t \geq b, \end{cases} \\ &= \mathbf{u}(t - a) - \mathbf{u}(t - b).\end{aligned}$$

Example: Write the piecewise defined function $f(t)$ in terms of pulse functions.

$$f(t) = \begin{cases} \sin t & 0 \leq t < \pi, \\ \cos t & \pi \leq t < 2\pi, \\ 0 & \text{else.} \end{cases}$$

Solution:

$$\begin{aligned}f(t) &= \sin(t) \begin{cases} 1 & 0 \leq t < \pi \\ 0 & \text{else} \end{cases} + \cos(t) \begin{cases} 1 & \pi \leq t < 2\pi, \\ 0 & \text{else} \end{cases} \\ &= \sin(t) \text{pulse}(t, 0, \pi) + \cos(t) \text{pulse}(t, \pi, 2\pi).\end{aligned}$$

Ramp Function

Definition:

$$\begin{aligned}\text{ramp}(t - a) &= \begin{cases} t - a & t \geq a, \\ 0 & t < a, \end{cases} \\ &= (t - a)\mathbf{u}(t - a).\end{aligned}$$

Example: Write the piecewise defined function $f(t)$ in terms of ramp functions.

$$f(t) = \begin{cases} 2t - 2 & t \geq 1, \\ 0 & t < 1. \end{cases}$$

Solution:

$$\begin{aligned}f(t) &= (2t - 2) \begin{cases} 1 & t \geq 1, \\ 0 & t < 1. \end{cases} \\ &= (2t - 2)\mathbf{u}(t - 1) \\ &= 2 \text{ramp}(t - 1).\end{aligned}$$

Piecewise Defined Functions

Definition:

$$f(t) = \begin{cases} f_1(t) & a_1 \leq t < a_2, \\ f_2(t) & a_2 \leq t < a_3, \\ \vdots & \vdots \\ f_n(t) & a_n \leq t < a_{n+1}, \\ 0 & \text{else} \end{cases}$$

The functions f_1, \dots, f_n must be defined on the whole real line.

Problem: Write the piecewise defined function $f(t)$ in terms of pulse functions.

Solution:

$$\begin{aligned} f(t) &= f_1(t) \begin{cases} 1 & a_1 \leq t < a_2, \\ 0 & \text{else} \end{cases} + \dots + f_n(t) \begin{cases} 1 & a_n \leq t < a_{n+1}, \\ 0 & \text{else} \end{cases} \\ &= f_1(t) \text{pulse}(t, a_1, a_2) + \dots + f_n(t) \text{pulse}(t, a_n, a_{n+1}). \end{aligned}$$

Laplace Pulse Example

$$f(t) = \begin{cases} e^{-t} & 1 \leq t < 2, \\ \cos(\pi t) & 2 \leq t < 3, \\ 0 & \text{else} \end{cases}$$

Problem: Find $L(f(t))$ by pulse decomposition.

Solution: We use $L(g(t)\mathbf{u}(t, a)) = e^{-as}L(g(t + a))$.

$$\begin{aligned} L(f(t)) &= L(e^{-t} \text{pulse}(t, 1, 2) + \cos(\pi t) \text{pulse}(t, 2, 3)) \\ &= L(e^{-t}\mathbf{u}(t - 1) - e^{-t}\mathbf{u}(t - 2) + \cos(\pi t)\mathbf{u}(t - 2)) - \\ &\quad L(\cos(\pi t)\mathbf{u}(t - 3)) \\ &= e^{-s}L(e^{-t-1}) - e^{-2s}L(e^{-t-2}) + e^{-2s}L(\cos(\pi t + 2\pi)) - \\ &\quad e^{-3s}L(\cos(\pi t + 3\pi)) \\ &= \frac{e^{-1-s} - e^{-2-2s}}{s + 1} + \frac{se^{-2s} - se^{-3s}}{s^2 + \pi^2}. \end{aligned}$$

Laplace of a Piecewise Defined Function

Definition:

$$f(t) = \begin{cases} f_1(t) & a_1 \leq t < a_2, \\ f_2(t) & a_2 \leq t < a_3, \\ \vdots & \vdots \\ f_n(t) & a_n \leq t < a_{n+1}, \\ 0 & \text{else} \end{cases}$$

Assumed is each f_j is defined on the whole real line.

Problem: Find $L(f(t))$ for the piecewise defined function $f(t)$.

Solution: We use $L(g(t)\mathbf{u}(t - a)) = e^{-as}L(g(t + a))$.

$$\begin{aligned} L(f(t)) &= L(f_1(t) \text{ pulse}(t, a_1, a_2)) + \cdots + L(f_n(t) \text{ pulse}(t, a_n, a_{n+1})) \\ &= \sum_{j=1}^n L(f_j(t)\mathbf{u}(t - a_j)) - L(f_j(t)\mathbf{u}(t - a_{j+1})) \\ &= \sum_{j=1}^n e^{-a_j s} L(f_j(t + a_j)) - e^{-a_{j+1} s} L(f_j(t + a_{j+1})). \end{aligned}$$