How to Solve Linear Differential Equations

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Euler Solution Atoms of Homogeneous Linear Differential Equations

Definition

- An Euler **base atom** is one of 1, $\cos bx$, $\sin bx$, e^{ax}, $e^{ax}\cos bx$, $e^{ax}\sin bx$, with $b > 0$ and $a \neq 0$.
- An Euler solution **atom** equals x^n times a base atom, where $n \geq 0$ is an integer.

Details and Remarks

- Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies that an atom is constructed from the complex expression $x^n e^{ax+ibx}$ by taking real and imaginary parts.
- The powers $1, x, x^2, \ldots, x^k$ are Euler solution atoms.
- The term that makes up an atom has coefficient 1, therefore $2e^x$ is not an atom, but the 2 can be stripped off to create the atom e^x. Zero is not an atom. Linear combinations like $2x + 3x^2$ are not atoms, but the individual terms x and x^2 are indeed atoms. Terms like $-e^x$, e^{-x^2}, $x^{5/2}\cos x$, $\ln|x|$ and $x/(1+x^2)$ are not atoms.

Independence

Linear algebra defines a list of functions $f_1, \ldots, f_k$ to be **linearly independent** if and only if the representation of the zero function as a linear combination of the listed functions is uniquely represented, that is,

$$0 = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) \text{ for all } x$$

implies $c_1 = c_2 = \cdots = c_k = 0$.

Independence and Atoms

Theorem 1 (Atoms are Independent)
A list of finitely many distinct Euler solution atoms is linearly independent.

Theorem 2 (Powers are Independent)
The list of distinct atoms $1, x, x^2, \ldots, x^k$ is linearly independent. And all of its sublists are linearly independent.

Construction of the General Solution from a List of Distinct Atoms

- **Picard's theorem** says that the homogeneous constant-coefficient linear differential equation

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0$$

  has solution space S of dimension n. Picard's theorem reduces the general solution problem to finding n linearly independent solutions.

- **Euler's theorem** *infra* says that the required n independent solutions can be selected as atoms. The theorem explains how to construct a list of distinct atoms, each of which is a solution of the differential equation, from the roots of the characteristic equation [Definition: The left side is the **characteristic polynomial**.]

$$r^n + p_{n-1}r^{n-1} + \cdots + p_1 r + p_0 = 0.$$

- The **Fundamental Theorem of Algebra** is part of the Doctoral thesis of Carl Friedrich Gauss (1777–1855): *An nth order polynomial equation has exactly n roots, real or complex, counted according to multiplicities*. Therefore, the characteristic equation has exactly n roots, counting multiplicities.

- **General Solution**. Because the list of atoms constructed by Euler's theorem has n distinct elements, then this list of independent atoms forms a **basis** for the general solution of the differential equation, giving

$$y = c_1(\text{atom } 1) + \cdots + c_n(\text{atom } n).$$

  Symbols $c_1, \ldots, c_n$ are arbitrary coefficients. In particular, each atom listed is itself a solution of the differential equation.

Euler's Basic Theorem

Theorem 3 (L. Euler)
The exponential $y = e^{r_1 x}$ is a solution of a constant-coefficient linear homogeneous differential of the nth order if and only if $r = r_1$ is a root of the characteristic equation.

- If $r_1 = a$ is a real root, then Euler's Theorem constructs *one real solution atom* e^{ax}.
- If $r_1 = a + ib$ is a complex root ($b > 0$), then Euler's Theorem constructs *two real solution atoms*
$$e^{ax}\cos bx, \quad e^{ax}\sin bx.$$

Derivation is from Euler's complex solution

$$e^{r_1 x} = e^{ax}\cos bx + ie^{ax}\sin bx.$$

The real and imaginary parts of this complex solution are real solutions of the differential equation. The conjugate pair of roots $a + ib$ and $a - ib$ produce the same two atoms, which explains the simplification above.

Euler's Multiplicity Theorem

Definition. A root $r = r_1$ of a polynomial equation $p(r) = 0$ has **multiplicity** k provided $(r - r_1)^k$ divides $p(r)$ but $(r - r_1)^{k+1}$ does not divide $p(r)$. The calculus equivalent is $\frac{d^j p}{dr^j}(r_1) = 0$ for $j = 0, .., k - 1$ and $\frac{d^k p}{dr^k}(r_1) \neq 0$.

Theorem 4 (L. Euler)
The expression $y = x^k e^{r_1 x}$ is a solution of a constant-coefficient linear homogeneous differential of the nth order if and only if $(r - r_1)^{k+1}$ divides the characteristic polynomial.

A Shortcut for using Euler's Theorems

Given a real root r_1 or complex root $r_1 = a + ib$, apply Euler's first theorem to obtain the base atom $e^{r_1 x}$, or the pair of base atoms $e^{ax}\cos bx, e^{ax}\sin bx$. Multiply each base item by powers 1, x, x^2, ..., until the number of atoms obtained equals the multiplicity of root r_1.

Atom List Examples

1. If root $r = -3$ has multiplicity 4, then the atom list is

$$e^{-3x}, xe^{-3x}, x^2e^{-3x}, x^3e^{-3x}.$$

The list is constructed by multiplying the base atom e^{-3x} by powers 1, x, x^2, x^3. The multiplicity 4 of the root equals the number of constructed atoms.

2. If $r = -3 + 2i$ is a root of the characteristic equation, then the base atoms for this root (both $-3 + 2i$ and $-3 - 2i$ counted) are

$$e^{-3x}\cos 2x, \quad e^{-3x}\sin 2x.$$

If root $r = -3 + 2i$ has multiplicity 3, then the two real atoms are multiplied by 1, x, x^2 to obtain a total of 6 atoms

$$\begin{array}{lll} e^{-3x}\cos 2x, & xe^{-3x}\cos 2x, & x^2e^{-3x}\cos 2x, \\ e^{-3x}\sin 2x, & xe^{-3x}\sin 2x, & x^2e^{-3x}\sin 2x. \end{array}$$

The number of atoms generated for each base atom is 3, which equals the multiplicity of the root $-3 + 2i$.

Theorem 5 (Homogeneous Solution y_h and Euler Solution Atoms)
Linear homogeneous nth order differential equations with constant coefficients have general solution $y_h(x)$ equal to a linear combination of n distinct atoms.

Theorem 6 (Particular Solution y_p and Euler Solution Atoms)
A linear non-homogeneous differential equation with constant coefficients a having forcing term $f(x)$ equal to a linear combination of atoms has a particular solution $y_p(x)$ which is a linear combination of atoms.

Theorem 7 (General Solution y and Euler Solution Atoms)
A linear non-homogeneous differential equation with constant coefficients having forcing term

$$f(x) = \text{ a linear combination of atoms}$$

has general solution

$$y(x) = y_h(x) + y_p(x) = \text{ a linear combination of atoms.}$$

Proofs

The first theorem follows from Picard's theorem, Euler's theorem and independence of atoms. The second follows from the method of undetermined coefficients, *infra*. The third theorem follows from the first two.