5.4 Heaviside's Method

This practical method was popularized by the English electrical engineer Oliver Heaviside (1850–1925). A typical application of the method is to solve

$$\frac{2s}{(s+1)(s^2+1)} = \mathcal{L}(f(t))$$

for the t-expression $f(t) = -e^{-t} + \cos t + \sin t$. The details in Heaviside's method involve a sequence of easy-to-learn college algebra steps.

More precisely, **Heaviside's method** systematically converts a polynomial quotient

$$\frac{a_0 + a_1 s + \cdots + a_n s^n}{b_0 + b_1 s + \cdots + b_m s^m} \tag{1}$$

into the form $\mathcal{L}(f(t))$ for some expression $f(t)$. It is assumed that $a_0, .., a_n, b_0, \ldots, b_m$ are constants and the polynomial quotient (1) has limit zero at $s = \infty$.

Partial Fraction Theory

In college algebra, it is shown that a rational function (1) can be expressed as the sum of **partial fractions**, which are terms of the form

$$\frac{A}{(s-s_0)^k}. \tag{2}$$

In (2), A is a real or complex constant and $(s-s_0)^k$ divides the denominator in (1). In particular, s_0 is a *root* of the denominator in (1).

Assume fraction (1) has **real coefficients**. If s_0 in (2) is real, then A is *real*. If $s_0 = \alpha + i\beta$ in (2) is *complex*, then $(s - \overline{s_0})^k$ also appears, where $\overline{s_0} = \alpha - i\beta$ is the complex conjugate of s_0. The corresponding terms in (2) turn out to be complex conjugates of one another, which can be combined in terms of *real* numbers B and C as

$$\frac{A}{(s-s_0)^k} + \frac{\overline{A}}{(s-\overline{s_0})^k} = \frac{B + C\,s}{((s-\alpha)^2 + \beta^2)^k}\,. \tag{3}$$

Simple Roots. Assume that (1) has *real coefficients* and the denominator of the fraction (1) has **distinct real roots** $s_1, \ldots, s_N$ and **distinct complex roots** $\alpha_1 \pm i\beta_1, \ldots, \alpha_M \pm i\beta_M$. The partial fraction expansion of (1) is a sum given in terms of *real* constants A_p, B_q, C_q by

$$\frac{a_0 + a_1 s + \cdots + a_n s^n}{b_0 + b_1 s + \cdots + b_m s^m} = \sum_{p=1}^{N} \frac{A_p}{s - s_p} + \sum_{q=1}^{M} \frac{B_q + C_q(s-\alpha_q)}{(s-\alpha_q)^2 + \beta_q^2}\,. \tag{4}$$

Multiple Roots. Assume (1) has *real coefficients* and the denominator of the fraction (1) has possibly **multiple roots**. Let N_p be the multiplicity of real root s_p and let M_q be the multiplicity of complex root $\alpha_q + i\beta_q$, $1 \le p \le N$, $1 \le q \le M$. The partial fraction expansion of (1) is given in terms of *real* constants $A_{p,k}$, $B_{q,k}$, $C_{q,k}$ by

$$(5) \qquad \sum_{p=1}^{N} \sum_{1\le k\le N_p} \frac{A_{p,k}}{(s-s_p)^k} + \sum_{q=1}^{M} \sum_{1\le k\le M_q} \frac{B_{q,k} + C_{q,k}(s-\alpha_q)}{((s-\alpha_q)^2+\beta_q^2)^k}.$$

A Failsafe Method

Consider the expansion in partial fractions

$$(6) \qquad \frac{s-1}{s(s+1)^2(s^2+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{Ds+E}{s^2+1}.$$

The five undetermined real constants A through E are found by **clearing the fractions**, that is, multiply (6) by the denominator on the left to obtain the polynomial equation

$$(7) \qquad \begin{aligned} s-1 &= A(s+1)^2(s^2+1) + Bs(s+1)(s^2+1) \\ &\quad +Cs(s^2+1) + (Ds+E)s(s+1)^2. \end{aligned}$$

Next, five different values of s are substituted into (7) to obtain equations for the five unknowns A through E. We always use the **roots of the denominator** to start: $s=0$, $s=-1$, $s=i$, $s=-i$ are the roots of $s(s+1)^2(s^2+1)=0$. Each complex root results in two equations, by taking real and imaginary parts. The complex conjugate root $s=-i$ is not used, because it duplicates the existing equation obtained from $s=i$. The three roots $s=0$, $s=-1$, $s=i$ give only four equations, so $s=1$ is used to get the fifth equation:

$$(8) \qquad \begin{aligned} -1 &= A & (s=0) \\ -2 &= -2C - 2(-D+E) & (s=-1) \\ i-1 &= (Di+E)i(i+1)^2 & (s=i) \\ 0 &= 8A+4B+2C+4(D+E) & (s=1) \end{aligned}$$

Because D and E are real, the complex equation ($s=i$) becomes two equations, as follows.

$i-1 = (Di+E)i(i^2+2i+1)$	Expand power.
$i-1 = -2Di - 2E$	Simplify using $i^2=-1$.
$1 = -2D$	Equate imaginary parts.
$-1 = -2E$	Equate real parts.

Solving the 5×5 system, the answers are $A=-1$, $B=2$, $C=0$, $D=-1/2$, $E=1/2$.

Heaviside's Coverup Method

The method applies only to the case of distinct roots of the denominator in (1). Extensions to multiple-root cases can be made; see page 233.

To illustrate Oliver Heaviside's ideas, consider the problem details

$$
\begin{aligned}
(9) \qquad \frac{2s+1}{s(s-1)(s+1)} &= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} \\
&= \mathcal{L}(A) + \mathcal{L}(Be^{t}) + \mathcal{L}(Ce^{-t}) \\
&= \mathcal{L}(A + Be^{t} + Ce^{-t})
\end{aligned}
$$

The first line (9) uses college algebra partial fractions. The second and third lines use the Laplace integral table and properties of $\mathcal{L}$.

Heaviside's mysterious method. Oliver Heaviside proposed to find in (9) the constant $C = -\frac{1}{2}$ by a **cover–up method**:

$$
\left.\frac{2s+1}{s(s-1)\boxed{}}\right|_{\boxed{s+1}=0} = \frac{C}{\boxed{}}.
$$

The *instructions* are to cover–up the matching factors $(s+1)$ on the left and right with box $\boxed{}$, then evaluate on the left at the *root* s which makes the contents of the box zero. The other terms on the right are replaced by zero.

To justify Heaviside's cover–up method, **clear the fraction** $C/(s+1)$, that is, multiply (9) by the denominator $\boxed{s+1}$ of the partial fraction $C/(s+1)$ to obtain:

$$
\frac{(2s+1)\boxed{(s+1)}}{s(s-1)\boxed{(s+1)}} = \frac{A\boxed{(s+1)}}{s} + \frac{B\boxed{(s+1)}}{s-1} + \frac{C\boxed{(s+1)}}{\boxed{(s+1)}}.
$$

Set $\boxed{(s+1)} = 0$ in the display. Cancellations left and right plus annihilation of two terms on the right gives Heaviside's prescription

$$
\left.\frac{2s+1}{s(s-1)}\right|_{s+1=0} = C.
$$

The factor $(s+1)$ in (9) is by no means special: the same procedure applies to find A and B. The method works for denominators with simple roots, that is, no repeated roots are allowed.

Extension to Multiple Roots. An extension of Heaviside's method is possible for the case of repeated roots. The basic idea is to *factor–out the repeats.* To illustrate, consider the partial fraction expansion details

$R = \dfrac{1}{(s+1)^2(s+2)}$	A sample rational function having repeated roots.
$= \dfrac{1}{s+1}\left(\dfrac{1}{(s+1)(s+2)}\right)$	Factor–out the repeats.
$= \dfrac{1}{s+1}\left(\dfrac{1}{s+1} + \dfrac{-1}{s+2}\right)$	Apply the cover–up method to the simple root fraction.
$= \dfrac{1}{(s+1)^2} + \dfrac{-1}{(s+1)(s+2)}$	Multiply.
$= \dfrac{1}{(s+1)^2} + \dfrac{-1}{s+1} + \dfrac{1}{s+2}$	Apply the cover–up method to the last fraction on the right.

Terms with only one root in the denominator are already partial fractions. Thus the work centers on expansion of quotients in which the denominator has two or more roots.

Special Methods. Heaviside's method has a useful extension for the case of roots of multiplicity two. To illustrate, consider these details:

$R = \dfrac{1}{(s+1)^2(s+2)}$	A fraction with multiple roots.
$= \dfrac{A}{s+1} + \dfrac{B}{(s+1)^2} + \dfrac{C}{s+2}$	See equation (5).
$= \dfrac{A}{s+1} + \dfrac{1}{(s+1)^2} + \dfrac{1}{s+2}$	Find B and C by Heaviside's cover–up method.
$= \dfrac{-1}{s+1} + \dfrac{1}{(s+1)^2} + \dfrac{1}{s+2}$	Multiply by $s+1$. Set $s = \infty$. Then $0 = A + 1$.

The illustration works for one root of multiplicity two, because $s = \infty$ will resolve the coefficient not found by the cover–up method.

In general, if the denominator in (1) has a root s_0 of multiplicity k, then the partial fraction expansion contains terms

$$\frac{A_1}{s-s_0} + \frac{A_2}{(s-s_0)^2} + \cdots + \frac{A_k}{(s-s_0)^k}.$$

Heaviside's cover–up method directly finds A_k, but not A_1 to A_{k-1}.