# Variation of Parameters

The initial value problem

$$y' + p(x)y = r(x), \quad y(x_0) = y_0, \tag{3}$$

where $p$ and $r$ are continuous in an interval containing $x = x_0$, has an explicit solution

$$y(x) = e^{-\int_{x_0}^{x} p(s)ds} \left( y_0 + \int_{x_0}^{x} r(t) e^{-\int_{x_0}^{t} p(s)ds} \, dt \right). \tag{4}$$

Formula (4) is called **variation of parameters**, for historical reasons.

While (4) has some appeal, applications use the **integrating factor method**, which is developed with indefinite integrals for computational efficiency. No one memorizes (4); they remember and study the *method*.

## Integrating Factor Identity

The technique called the **integrating factor method** uses the replacement rule

$$\text{The fraction} \quad \frac{\left(e^{\int p(x)dx} Y\right)'}{e^{\int p(x)dx}} \quad \text{replaces} \quad Y' + p(x)Y. \qquad (5)$$

The factor $e^{\int p(x)dx}$ in (5) is called an **integrating factor**.

# The Integrating Factor Method

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| **Standard Form** | Rewrite $y' = f(x, y)$ in the form $y' + p(x)y = r(x)$ where $p$, $r$ are continuous. The method applies only in case this is possible. |
| **Find $W$** | Find a simplified formula for $W = e^{\int p(x)dx}$. The antiderivative $\int p(x)dx$ can be chosen conveniently. |
| **Prepare for Quadrature** | Obtain the new equation $\frac{(Wy)'}{W} = r$ by replacing the left side of $y' + p(x)y = r(x)$ by equivalence (5). |
| **Method of Quadrature** | Clear fractions to obtain $(Wy)' = rW$. Apply the method of quadrature to get $Wy = \int r(x)W(x)dx + C$. Divide by $W$ to isolate the explicit solution $y(x)$. |

Equation (5) is central to the method, because it collapses the two terms $y' + py$ into a single term $(Wy)'/W$; the method of quadrature applies to $(Wy)' = rW$. The literature calls the exponential factor $W$ an **integrating factor** and equivalence (5) a **factorization** of $Y' + p(x)Y$.