Elementary Matrices and Frame Sequences

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Elementary Matrices

**Definition.** An elementary matrix $E$ is the result of applying a combination, multiply or swap rule to the identity matrix.

An elementary matrix is then the **second frame** after a combo, swap or mult toolkit operation which has been applied to a **first frame** equal to the identity matrix.

**Example:**

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]  
First frame = identity matrix.

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-5 & 0 & 1 \\
\end{pmatrix}
\]  
Second frame  
Elementary combo matrix  
combo(1, 3, -5)
Computer algebra systems and elementary matrices

The computer algebra system Maple displays typical $4 \times 4$ elementary matrices (C=Combination, M=Multiply, S=Swap) as follows.

```
with(linalg):
Id:=diag(1,1,1,1);
C:=addrow(Id,2,3,c);
M:=mulrow(Id,3,m);
S:=swaprow(Id,1,4);
```

The answers:

$$
C = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & c & 1 & 0 \\
0 & 0 & 0 & 1 
\end{pmatrix}, \quad M = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & m & 0 \\
0 & 0 & 0 & 1 
\end{pmatrix},
$$

$$
S = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 
\end{pmatrix}.
$$
Constructing elementary matrices $E$ and their inverses $E^{-1}$

**Mult** Change a one in the identity matrix to symbol $m \neq 0$.

**Combo** Change a zero in the identity matrix to symbol $c$.

**Swap** Interchange two rows of the identity matrix.

Constructing $E^{-1}$ from elementary matrix $E$

**Mult** Change diagonal multiplier $m \neq 0$ in $E$ to $1/m$.

**Combo** Change multiplier $c$ in $E$ to $-c$.

**Swap** The inverse of $E$ is $E$ itself.
Theorem 1 (Frame sequences and elementary matrices)
In a frame sequence, let the second frame $A_2$ be obtained from the first frame $A_1$ by a _combo_, _swap_ or _mult_ toolkit operation. Let $n$ equal the row dimension of $A_1$. Then there is correspondingly an $n \times n$ _combo_, _swap_ or _mult_ elementary matrix $E$ such that

$$A_2 = EA_1.$$  

Theorem 2 (The $\text{rref}$ and elementary matrices)
Let $A$ be a given matrix of row dimension $n$. Then there exist $n \times n$ elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$\text{rref}(A) = E_k \cdots E_2 E_1 A.$$
Proof of Theorem 1
The first result is the observation that left multiplication of matrix $A_1$ by elementary matrix $E$ gives the answer $A_2 = EA_1$ which is obtained by applying the corresponding combo, swap or mult toolkit operation. This fact is discovered by doing examples, then a formal proof can be constructed (not presented here).

Proof of Theorem 2
The second result applies the first result multiple times to obtain elementary matrices $E_1$, $E_2$, ... which represent the multiply, combination and swap operations performed in the frame sequence which take the First Frame $A_1 = A$ into the Last Frame $A_{k+1} = \text{rref}(A_1)$. Combining the identities

$$A_2 = E_1 A_1, \quad A_3 = E_2 A_2, \quad \ldots, \quad A_{k+1} = E_k A_k$$

gives the matrix multiply equation

$$A_{k+1} = E_k E_{k-1} \cdots E_2 E_1 A_1$$

or equivalently the theorem’s result, because $A_{k+1} = \text{rref}(A)$ and $A_1 = A$. 
A certain 6-frame sequence.

\[
\begin{align*}
A_1 &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 6 & 3 \end{pmatrix} \quad \text{Frame 1, original matrix.} \\
A_2 &= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -6 \\ 3 & 6 & 3 \end{pmatrix} \quad \text{Frame 2, combo(1,2,-2).} \\
A_3 &= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 3 & 6 & 3 \end{pmatrix} \quad \text{Frame 3, mult(2,-1/6).} \\
A_4 &= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -6 \end{pmatrix} \quad \text{Frame 4, combo(1,3,-3).} \\
A_5 &= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Frame 5, combo(2,3,-6).} \\
A_6 &= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Frame 6, combo(2,1,-3). Found \text{ rref}(A_1).} 
\end{align*}
\]
The corresponding $3 \times 3$ elementary matrices are

\[
E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Frame 2, combo(1,2,-2) applied to } I.
\]

\[
E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Frame 3, mult(2,-1/6) applied to } I.
\]

\[
E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \quad \text{Frame 4, combo(1,3,-3) applied to } I.
\]

\[
E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix} \quad \text{Frame 5, combo(2,3,-6) applied to } I.
\]

\[
E_5 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Frame 6, combo(2,1,-3) applied to } I.
\]
Frame Sequence Details

\[
\begin{align*}
A_2 &= E_1 A_1 & \text{Frame 2, } E_1 \text{ equals combo}(1,2,-2) \text{ on } I. \\
A_3 &= E_2 A_2 & \text{Frame 3, } E_2 \text{ equals mult}(2,-1/6) \text{ on } I. \\
A_4 &= E_3 A_3 & \text{Frame 4, } E_3 \text{ equals combo}(1,3,-3) \text{ on } I. \\
A_5 &= E_4 A_4 & \text{Frame 5, } E_4 \text{ equals combo}(2,3,-6) \text{ on } I. \\
A_6 &= E_5 A_5 & \text{Frame 6, } E_5 \text{ equals combo}(2,1,-3) \text{ on } I. \\
A_6 &= E_5 E_4 E_3 E_2 E_1 A_1 & \text{Summary frames 1-6.}
\end{align*}
\]

Then

\[
\text{rref}(A_1) = E_5 E_4 E_3 E_2 E_1 A_1,
\]

which is the result of the Theorem.
Fundamental Theorem Illustrated

The summary:

\[ A_6 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A_1 \]

Because \( A_6 = \text{rref}(A_1) \), the above equation gives the inverse relationship

\[ A_1 = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} \text{rref}(A_1). \]

Each inverse matrix is simplified by the rules for constructing \( E^{-1} \) from elementary matrix \( E \), the result being

\[ A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{rref}(A_1) \]
Theorem 3 (RREF Inverse Method)

\[ \text{rref}(\text{aug}(A, I)) = \text{aug}(I, B) \text{ if and only if } AB = I. \]

**Proof:** For any matrix \( E \) there is the matrix multiply identity

\[ E \text{ aug}(C, D) = \text{aug}(EC, ED). \]

This identity is proved by arguing that each side has identical columns. For example, \( \text{col}(\text{LHS}, 1) = E \text{ col}(C, 1) = \text{col}(\text{RHS}, 1). \)

Assume \( C = \text{aug}(A, I) \) satisfies \( \text{rref}(C) = \text{aug}(I, B) \). The fundamental theorem of elementary matrices implies \( E_k \cdots E_1 C = \text{rref}(C) \). Then

\[ \text{rref}(C) = \text{aug}(E_k \cdots E_1 A, E_k \cdots E_1 I) = \text{aug}(I, B) \]

implies that \( E_k \cdots E_1 A = I \) and \( E_k \cdots E_1 I = B \). Together, \( BA = I \) and then \( B \) is the inverse of \( A \).

Conversely, assume that \( AB = I \). Then \( A \) has inverse \( B \). The fundamental theorem of elementary matrices implies the identity \( E_k \cdots E_1 A = \text{rref}(A) = I \). It follows that \( B = E_k \cdots E_1 \). Then \( \text{rref}(C) = E_k \cdots E_1 \text{ aug}(A, I) = \text{aug}(E_k \cdots E_1 A, E_k \cdots E_1 I) = \text{aug}(I, B). \)