Differential Equations and Linear Algebra

2250-1 at 7:30am on 29 April 2013

Instructions. The time allowed is 120 minutes. The examination consists of eight problems, one for each of chapters 3, 4, 5, 6, 7, 8, 9, 10, each problem with multiple parts. A chapter represents 15 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c), $\cdots$. Each chapter (3 to 10) adds at most 100 towards the maximum final exam score of 800. The final exam grade is reported as a percentage 0 to 100, as follows:

$$\text{Final Exam Grade} = \frac{\text{Sum of scores on eight chapters}}{8}.$$

- Calculators, books, notes, computers and electronics are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader. More credit is possible if you write something.
- Answer checks are not expected and they are not required. First drafts are expected, not complete presentations.
- Please prepare **exactly one** stapled package of all eight chapters, organized by chapter. All scratch work for a chapter must appear in order. Any work stapled out of order could be missed, due to multiple graders.
- The graded exams will be in a box outside 113 JWB; you will pick up one stapled package.
- Records will be posted on **CANVAS**, found from the Registrar's web site link. Recording errors can be reported by email, hopefully as soon as discovered, but also even weeks after grades are posted.

Final Grade. The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90 + 91 + 92 + 89 + 89}{5} = 90.2.$$

Dailies, both maple and math together, count 30% of the final grade. The course average is computed from the formula

$$\text{Course Average} = \frac{70}{100}(\text{Exam Average}) + \frac{30}{100}(\text{Dailies Average}).$$

Please recycle this page or keep it for your records.

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Mathematics 2250-1 Final Exam at 7:30am on 29 April 2013

Scores
Ch3.
Ch4.
Ch5.
Ch6.
Ch7.
Ch8.
Ch9.
Ch10.

Ch3. (Linear Systems and Matrices) Complete all problems.

[40%] **Ch3(a)**: Incorrect answers lose all credit. **Circle the correct answer**.
True or False: **Ch3(a) Part 1**. [10%]:
Assume given a 3×3 matrix A and invertible 3×3 elementary matrices E_1, E_2, E_3. Define $U = E_3E_2E_1A$. Then U is upper triangular, but U is not always invertible.
True or False: **Ch3(a) Part 2**. [10%]:
If a 3×3 matrix A has no inverse, then there always exists a nonzero vector $\vec{b}$ such that the equation $A\vec{x} = \vec{b}$ has at least one solution $\vec{x}$.
True or False: **Ch3(a) Part 3**. [10%]:
An invertible $n \times n$ matrix A can be written as the product of elementary matrices.
True or False: **Ch3(a) Part 4**. [10%]:
Given a 3×1 matrix A and a 3×1 vector $\vec{b}$, then the system $A\vec{x} = \vec{b}$ can be solved for $\vec{x}$, provided $\vec{b}$ is a linear combination of the columns of the transpose A^T.

Answer: False. False. True. False.

[20%] **Ch3(b)**: Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 \\ 0 & -4 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}.$$

For the system $A\vec{x} = \vec{b}$, find x_1, x_2 by Cramer's Rule, showing **all details** (details count 75%).

Answer: $x_1 = \Delta_1/\Delta$, $x_2 = \Delta_2/\Delta$, $\Delta = \det\begin{pmatrix} -2 & 3 \\ 0 & -4 \end{pmatrix} = 8$, $\Delta_1 = \det\begin{pmatrix} -3 & 3 \\ 5 & -4 \end{pmatrix} = -3$, $\Delta_2 = \det\begin{pmatrix} -2 & -3 \\ 0 & 5 \end{pmatrix} = -10$, $x_1 = \frac{-3}{8}$, $x_2 = \frac{-10}{8} = \frac{-5}{4}$.

[40%] **Ch3(c)**: Determine which values of k correspond to infinitely many solutions for the system $A\vec{x} = \vec{b}$ given by

$$A = \begin{pmatrix} 0 & 4 & k \\ 0 & 2-k & 2-k \\ 1 & 4 & 5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ -1/2 \\ k \end{pmatrix}.$$

Answer: (1) There is a unique solution for $\det(A) = k^2 - 6k + 8 \neq 0$, which implies $k \neq 2$ and $k \neq 4$. Therefore, the answer for infinitely many solutions, if there is one, must have either $k = 2$ or $k = 4$ or both. Elimination methods applied to the augmented matrix $C =< A|\vec{b} >$, using two swaps and one combo, give the frame $\begin{pmatrix} 1 & 4 & 5 & 1 \\ 0 & 4 & k & 1 \\ 0 & 0 & A & B \end{pmatrix}$, where $A = (2-k)(4-k)/4$ and $B = (k-4)/4$. The no solution case must have a signal equation. The infinite solution case must have a free variable but no signal equation. Then (2) No solution for $k = 2$ [equation 2 is a signal equation]; (3) Infinitely many solutions for $k = 4$ [a free variable x_3 but no signal equation].

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Ch4. (Vector Spaces) Complete all problems.

[30%] **Ch4(a)**: Check the independence tests which apply to prove that vectors $x, x^{7/3}, e^x$ are independent in the vector space of all continuous functions on $-\infty < x < \infty$.

☐	**Wronskian test**	Wronskian of functions f, g, h nonzero at $x = x_0$ implies independence of f, g, h.
☐	**Rank test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Atom test**	Any finite set of distinct Euler solution atoms is independent.
☐	**Pivot test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.
☐	**Sampling test**	Let samples a, b, c be given and for functions f, g, h define $A = \begin{pmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f(c) & g(c) & h(c) \end{pmatrix}$. Then $\det(A) \neq 0$ implies independence of f, g, h.

Answer: Tests 2, 3, 5 fail to apply, because these tests are about fixed vectors, not functions. Details for the other tests can be given: let $f(x) = x, g(x) = x^{7/3}, h(x) = e^x$. These are not atoms, so the atom test does not apply. The Wronskian test applies directly, using $x_0 = 1$ to obtain Wronskian determinant value $W = 4e/3$. The sampling test applies using samples $a = 0$, $b = 1$, $c = 2$ because then $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & e \\ 2 & 4\sqrt[3]{2} & e^2 \end{pmatrix}$ and $\det(A) \neq 0$.

[20%] **Ch4(b)**: Consider the homogenous system $A\vec{x} = \vec{0}$. The nullity of A equals the number of free variables. Give an example of a matrix A with three pivot columns that has nullity 2.

Answer: Let $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ be any three columns of the identity and let $\vec{v}_4 = \vec{v}_5$ each be the zero vector. Define A to be the augmented matrix of these four vectors. Then A has 5 columns. There are three pivot columns and two free variables x_4, x_5, hence A has 3 pivots and nullity 2.

[30%] **Ch4(c)**: Let V be the vector space of all continuously differentiable vector functions $\vec{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Let S be the set of all vector solutions $\vec{v}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ of the dynamical system

$$\begin{cases} x'(t) = 2x(t) \\ y'(t) = 4y(t) \end{cases}$$

Find two independent solutions $\vec{v}_1, \vec{v}_2$ such that $S = \mathbf{span}(\vec{v}_1, \vec{v}_2)$. *This calculation proves that S is a subspace of V by Picard's theorem and the Span Theorem, hence S is a vector space.*

Answer: The dynamical system is diagonal, therefore it can be solved by the method of linear cascades (Section 1.5). The general solution is $x = c_1 e^{2t}$, $y = c_2 e^{4t}$. Then two independent vector solutions are found by taking partial derivatives on the symbols c_1, c_2, obtaining $\vec{v}_1 = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ e^{4t} \end{pmatrix}$. Then any solution of the dynamical system is given by $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$.

[20%] **Ch4(d)**: The 4×6 matrix A below has some independent columns. Report the independent columns of A, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 6 & 6 & 0 & 0 & 3 \end{pmatrix}$$

Answer: Find $\mathbf{rref}(A) = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. The pivot columns are 2 and 3.

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Ch5. (Linear Equations of Higher Order) Complete all problems.

[20%] **Ch5(a)**: Find the characteristic equation of a higher order linear homogeneous differential equation with constant coefficients, of minimum order, such that $y = 3x^2 + 10xe^{-x} + 4\cos(2x)$ is a solution.

Answer: The atoms x^2, xe^{-x}, $\cos(2x)$ correspond to roots $0, 0, 0, -1, -1, 2i, -2i$. The factor theorem implies the characteristic polynomial should be $r^3(r+1)^2(r^2+4)$.

[20%] **Ch5(b)**: Determine a *basis of solutions* of a homogeneous constant-coefficient linear differential equation, given it has characteristic equation

$$(r^4 - 4r^3)((r - \ln(2))^2 + 4)^2 = 0.$$

Answer: The roots are $0, 0, 0, 4, \ln(2) \pm 2i, \ln(2) \pm 2i$. By Euler's theorem, a basis is the set of atoms for these roots: $1, x, x^2, e^{4x}, e^{\ln(2)x}\cos(2x), e^{\ln(2)x}\sin(2x), xe^{\ln(2)x}\cos(2x), xe^{\ln(2)x}\sin(2x)$. The exponential factor $e^{\ln(2)x}$ can be written 2^x.

[30%] **Ch5(c)**: Find the **Beats** solution for the forced undamped spring-mass problem

$$x'' + 64x = 40\cos(4t), \quad x(0) = x'(0) = 0.$$

It is known that this solution is the sum of two harmonic oscillations of different frequencies.

Answer: Use undetermined coefficients trial solution $x = d_1 \cos 4t + d_2 \sin 4t$. Then $d_1 = 5/6$, $d_2 = 0$.Then $x_p(t) = (5/6)\cos(4t)$. The characteristic equation $r^2 + 64 = 0$ has roots $\pm 8i$ and the corresponding Euler solution atoms are $\cos(8t), \sin(8t)$. Then $x_h(t) = c_1\cos(8t) + c_2\sin(8t)$. The general solution is $x = x_h + x_p$. Now use $x(0) = x'(0) = 0$ to determine $c_1 = -5/6, c_2 = 0$. Then $x(t) = -(5/6)\cos(8t) + (5/6)\cos(4t)$.

[30%] **Ch5(d)**: Determine the **shortest** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$\frac{d^4y}{dx^4} - 4\frac{d^2y}{dx^2} = 11x^2 + 2x + 3 + 12\cos 2x + 13xe^{2x}$$

Answer: The homogeneous problem has roots $0, 0, \pm 2i$ with atoms $1, x, e^{2x}, e^{-2x}$. The trial solution is constructed initially from $f(x) = 11x^2 + 2x + 3 + 12\cos 2x + 13xe^{2x}$, which has seven atoms in a list in four groups (1) $1, x, x^2$; (2) $\cos 2x$; (3) $\sin 2x$; (4) e^{2x}, xe^{2x}. Conflicts with the homogeneous equation atoms causes a repair of groups (1), (4) making the new groups (1) x^2, x^3, x^4; (2) $\cos 2x$; (3) $\sin 2x$; (4) xe^{2x}, x^2e^{2x}. Then the shortest trial solution is a linear combination of the seven atoms in the corrected list.

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Ch6. (Eigenvalues and Eigenvectors) Complete all problems.

[30%] **Ch6(a)**: Let C be a 2×2 matrix having eigenpairs

$$\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(-1, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right).$$

Find the matrix C.

Answer: Write $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$. Then $CP = PD$ implies $C = PDP^{-1} = \begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}$.

[30%] **Ch6(b)**: Find the eigenvalues of the matrix B:

$$B = \begin{pmatrix} 2 & 4 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{pmatrix}$$

Answer: The characteristic polynomial is $\det(B - rI) = (2-r)(5-r)(4-r)(2-r)$. The eigenvalues are $2, 2, 4, 5$.

It is possible to directly find the eigenvalues of B by cofactor expansion of $|B - rI|$. This is the expected method. Expand $|B - rI|$ by cofactors on row 2. The calculation reduces to a 3×3 determinant. Expand the 3×3 along row 1 to reduce the calculation to a 2×2, which evaluates as a quadratic by Sarrus' Rule.

An alternate method is described below, which depends upon a determinant product theorem for special block matrices, such as encountered in this example.

Matrix B is a block matrix $B = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline 0 & B_3 \end{array}\right)$, where B_1, B_2, B_3 are all 2×2 matrices. Then $B - rI = \left(\begin{array}{c|c} B_1 - rI & B_2 \\ \hline 0 & B_3 - rI \end{array}\right)$. Using the determinant product theorem for such special block matrices (zero in the left lower block) gives $|B - rI| = |B_1 - rI||B_3 - rI|$. Because a determinant equals the determinant of its transpose, then the answer is that B has eigenvalues equal to the eigenvalues of B_1 and B_3, even if the zero block is in the upper right. The two 2×2 determinants are quickly found by Sarrus' Rule: $|B_1 - rI| = (2-r)(5-r)$ and $|B_3 - rI| = r^2 - 6r + 8 = (4-r)(2-r)$.

[40%] **Ch6(c)**: The matrix $A = \begin{pmatrix} 1 & -4 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is **not diagonalizable**. Display the details for computing all the eigenpairs. *Details count 75%.*

Answer: The characteristic equation is $(3-r)^3 = 0$ with roots $3, 3, 3$. Because $A - 3I = \begin{pmatrix} -2 & -4 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has reduced row echelon form $B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then there are two eigenpairs for $\lambda = 3$, with eigenvectors $\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$.

Place this page on top of all Ch6 work.

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Ch7. (Linear Systems of Differential Equations) Complete all problems.

[30%] **Ch7(a)**: Incorrect answers lose all credit. **Circle the correct answer**.
True or False: **Ch7(a) Part 1**. [10%]:
A linear dynamical system $\frac{d}{dt}\vec{u} = A\vec{u}$ cannot always be solved by Laplace theory methods, because the matrix A may fail to be diagonalizable.
True or False: **Ch7(a) Part 2**. [10%]:
A linear dynamical system $\frac{d}{dt}\vec{u} = A\vec{u}$ has solution $\vec{u}(t) = e^{At}\vec{c}$, where $\vec{c}$ is a vector which represents the state variable $\vec{u}(t)$ at instant $t = 0$. The result requires A to be diagonalizable.
True or False: **Ch7(a) Part 3**. [10%]:
Linear dynamical systems $\frac{d}{dt}\vec{u} = A\vec{u}$ exist with A of size 10×10 which can be solved by the linear integrating factor method.

Answer: False. False. True.

[30%] **Ch7(b)**: Solve the 3×3 linear dynamical system $\vec{u}' = A\vec{u}$, given matrix

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Use the most efficient method possible.

Answer: The matrix is triangular, so the theory of linear cascades applies. First, $x_3' = 2x_3$, then $x_3 = c_1 e^{2t}$. Back-substitute: $x_2' = 2x_3 = 2c_1 e^{2t}$ implies $x_2 = c_1 e^{2t} + c_2$. Then $x_1' = 2x_2 = 2c_1 e^{2t} + 2c_2$, which implies $x_1 = c_1 e^{2t} + 2c_2 t + c_3$. Some fundamental matrices, obtained from this method:

$$\Phi(t) = \begin{pmatrix} e^{2t} & 2t & 1 \\ e^{2t} & 1 & 0 \\ e^{2t} & 0 & 0 \end{pmatrix}, \Psi(t) = \begin{pmatrix} 1? & t & e^{2t} \\ 1 & 1/2 & e^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix}.$$

Does Cayley-Hamilton-Ziebur work? Yes, it always works. The roots of $|A - rI| = 0$ are $r = 0, 0, 2$, which implies atoms $1, t, e^{2t}$ and then $\vec{u} = \vec{d}_1 + \vec{d}_2 t + \vec{d}_3 e^{2t}$. Differentiate this relation two times and set $t = 0$ in the resulting 3 equations in 3 vector unknowns $\vec{d}_1, \vec{d}_2, \vec{d}_3$. Solve the system for the unknown vectors, in terms of A and $\vec{u}(0) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \equiv \vec{c}$. You will get $\vec{d}_3 = \frac{1}{4}A^2\vec{c}$, $\vec{d}_2 = A\vec{c} - \frac{1}{2}A^2\vec{c}$, $\vec{d}_1 = (I - \frac{1}{4}A^2)\vec{c}$. Then $\vec{u}(t) = \left(I - \frac{1}{4}A^2 + (A - \frac{1}{2}A^2)t + \frac{1}{4}A^2 e^{2t}\right)\vec{c}$. This answer is slightly different than what was obtained by linear cascades, and it takes more time to produce.

Does the Eigenanalysis method apply? The roots of the characteristic equation are $0, 0, 2$, and the corresponding atoms are $1, t, e^{2t}$. But $A - 0I$ is just A, which has rank 2 and nullity 1. There is only one eigenpair: A is not diagonalizable. The eigenanalysis method *does not apply*.

Laplace theory? Yes, it always works. Begin with the Laplace resolvent equation $(sI - A)\mathcal{L}(\vec{u}) = \vec{u}(0) = \vec{c}$. Solve as $\mathcal{L}(\vec{u}) = (sI - A)^{-1}\vec{c}$. Using the backward Laplace table on each entry in $(sI - A)^{-1}$ gives the answer $\vec{u}(t) = \begin{pmatrix} 1 & t & a(t) \\ 0 & 1 & b(t)\,t \\ 0 & 0 & c(t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, where $a(t) = -t + (e^{2t} - 1)/2, b(t) = e^{2t} - 1, c(t) = e^{2t}$. None of this is easy, and the preferred method is linear cascades.

Matrix exponential? Yes, it always gives the solution $\vec{u}(t) = e^{At}\vec{c}$, where $\vec{c}$ is a column vector of arbitrary constants. Laplace theory finds $e^{At} = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)$, identical to the matrix found above for Laplace theory, $e^{At} = \begin{pmatrix} 1 & t & a(t) \\ 0 & 1 & b(t)\,t \\ 0 & 0 & c(t) \end{pmatrix}$. This same matrix can be found using Putzer's formula for the 3×3 case.

[40%] **Ch7(c)**: Apply the eigenanalysis method to solve the system $\frac{d}{dt}\vec{u} = A\vec{u}$, when the matrix is defined as

$$A = \begin{pmatrix} -7 & 1 & 1 \\ 1 & -7 & 1 \\ 0 & 0 & -7 \end{pmatrix}$$

Answer: The eigenpairs of A are $(-8, \vec{v}_1), (-7, \vec{v}_2), (-6, \vec{v}_3)$ where

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Then

$$\vec{u}(t) = c_1 e^{-8t}\vec{v}_1 + c_1 e^{-7t}\vec{v}_2 + c_1 e^{-6t}\vec{v}_3.$$

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Ch8. (Matrix Exponential) Complete all problems.

[40%] **Ch8(a)**: Using any method in the lectures or the textbook, display the matrix exponential e^{Bt}, for

$$B = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}.$$

Answer: The exponential matrix is $\begin{pmatrix} e^t & 0 \\ 2e^t - 2 & 1 \end{pmatrix}$.

Method 1. We find a fundamental matrix $Z(t)$. It is done by solving the matrix system $\vec{u}' = B\vec{u}$ with the method of linear cascades. Then $x = c_1 e^t$, $y = c_2 + 2c_1 e^t$. Write $\vec{u} = Z(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ where $Z(t) = \begin{pmatrix} e^t & 0 \\ 2e^t & 1 \end{pmatrix}$. Then

$$e^{Bt} = Z(t)Z(0)^{-1} = \begin{pmatrix} e^t & 0 \\ 2e^t & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} e^t & 0 \\ 2e^t - 2 & 1 \end{pmatrix}.$$

Method 2. Putzer's formula can be used, $e^{Bt} = e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(B - \lambda_1 I)$. The eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$ are found from $\det(B - \lambda I) = \lambda^2 - \lambda = 0$. Then Putzer's formula becomes $e^{Bt} = I + \frac{1-e^t}{0-1}(B - 0I)$. Expanding gives the reported answer.

Method 3. A third method would use the formula $e^{Bt} = \mathcal{L}^{-1}((sI - B)^{-1})$ to find the answer in a sequence of Laplace steps. This method is certainly longer.

[30%] **Ch8(b)**: Consider the 2×2 system

$$\begin{aligned} x' &= -2x, \\ y' &= 3y, \\ x(0) &= 1, \quad y(0) = 2. \end{aligned}$$

Solve the system as a matrix problem $\frac{d}{dt}\vec{u} = A\vec{u}$ for $\vec{u}$, using the matrix exponential e^{At}.

Answer: Let $A = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$. The answer for e^{At} can be obtained quickly from the theorem $e^{\mathbf{diag}(a,b)t} = \mathbf{diag}(e^{at}, e^{bt})$, giving the answer $e^{At} = \mathbf{diag}(e^{-2t}, e^{3t})$. Then use $\vec{u}(t) = e^{At}\vec{u}(0)$, which implies $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{-2t} \\ 2e^{3t} \end{pmatrix}$.

[30%] **Ch8(c)**: Display the matrix form of variation of parameters for the 2×2 system. Then integrate to find one particular solution.

$$\begin{aligned} x' &= -2x + 4, \\ y' &= 3y. \end{aligned}$$

Answer: Variation of parameters is $\vec{u}_p(t) = e^{At}\int_0^t e^{-As}\begin{pmatrix}4\\0\end{pmatrix}ds$. Then $e^{At} = \mathbf{diag}(e^{-2t}, e^{3t})$ from the previous problem. Substitute $t \to -s$ to obtain $e^{-As} = \mathbf{diag}(e^{2s}, e^{-3s})$. The integration step is

$$\int_0^t \mathbf{diag}(e^{2s}, e^{-3s})\begin{pmatrix}4\\0\end{pmatrix}ds = \int_0^t \begin{pmatrix}4e^{2s}\\0\end{pmatrix}ds = \begin{pmatrix}2e^{2t}-2\\0\end{pmatrix}.$$

Finally, $\vec{u}_p(t) = \mathbf{diag}(e^{-2t}, e^{3t})\begin{pmatrix}2e^{2t}-2\\0\end{pmatrix} = \begin{pmatrix}2-2e^{-2t}\\0\end{pmatrix}$.

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Ch9. (Nonlinear Systems) Complete all problems.

[30%] **Ch9(a)**:
Determine whether the unique equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node.

$$\vec{u}' = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} \vec{u}$$

Answer: It is an unstable spiral. Details: The eigenvalues of A are roots of $r^2 + 2r + 5 = (r+1)^2 + 4 = 0$, which are complex conjugate roots $1 \pm 2i$. Rotation eliminates the saddle and node. Finally, the atoms $e^t \cos 2t$, $e^t \sin 2t$ have limit zero at $t = -\infty$, therefore the system is stable at $t = -\infty$ and unstable at $t = \infty$. So it must be a spiral [centers have no exponentials]. Report: **unstable spiral**.

[30%] **Ch9(b)**: Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - 2y^2 - y + 32, \\ y' &= 2x^2 - 2xy. \end{aligned}$$

An equilibrium point is $x = 4$, $y = 4$. Compute the Jacobian matrix $A = J(4, 4)$ of the linearized system at this equilibrium point.

Answer: The Jacobian is $J(x, y) = \begin{pmatrix} 1 & -4y - 1 \\ 4x - 2y & -2x \end{pmatrix}$. Then $A = J(4, 4) = \begin{pmatrix} 1 & -17 \\ 8 & -8 \end{pmatrix}$.

[40%] **Ch9(c)**: Consider the nonlinear dynamical system

$$\begin{aligned} x' &= -4x + 4y + 9 - x^2, \\ y' &= 3x - 3y. \end{aligned}$$

At equilibrium point $x = 3$, $y = 3$, the Jacobian matrix is $A = J(3, 3) = \begin{pmatrix} -10 & 4 \\ 3 & -3 \end{pmatrix}$.

(1) Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the linear system $\frac{d}{dt}\vec{u} = A\vec{u}$.

(2) Apply a theorem to classify $x = 3$, $y = 3$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. *Details count 75%.*

Answer: (1) The Jacobian is $J(x, y) = \begin{pmatrix} -4 - 2x & 4 \\ 3 & -3 \end{pmatrix}$. Then $A = J(3, 3) = \begin{pmatrix} -10 & 4 \\ 3 & -3 \end{pmatrix}$.
The eigenvalues of A are found from $r^2 + 13r + 18 = 0$, giving distinct real negative roots $-\frac{13}{2} \pm (\frac{1}{2})\sqrt{97}$. Because there are no trig functions in the Euler solution atoms, then no rotation happens, and the classification must be a saddle or node. The Euler solution atoms limit to zero at $t = \infty$, therefore it is a node and we report a **stable node** for the linear problem $\vec{u}' = A\vec{u}$ at equilibrium $\vec{u} = \vec{0}$. (2) Theorem 2 in Edwards-Penney section 9.2 applies to say that the same is true for the nonlinear system: **stable node** at $x = 3$, $y = 3$. The exceptional case in Theorem 2 is a proper node, having characteristic equation roots that are equal. Stability is always preserved for nodes.

Place this page on top of all Ch9 work.

Name ______________________________________ **29Apr2013**

Mathematics 2250-1 Final Exam at 7:30am on 29 April 2013

Ch10. (Laplace Transform Methods) Complete all problems.

It is assumed that you know the minimum forward Laplace integral table and the 8 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

[40%] **Ch10(a)**: Fill in the blank spaces in the Laplace tables. Each wrong answer subtracts 3 points from the total of 40.

$f(t)$					
$\mathcal{L}(f(t))$	$\frac{113}{s^3}$	$\frac{3}{3s+4}$	$\frac{-8}{s^2+16}$	$\frac{\pi s}{s^2+4}$	$\frac{e^{-2s}}{s+1}$

$f(t)$	t	$(t+e^{-2t})e^t$	$e^t\cos(2t)$	t^3e^{-t}	$t\cos t$
$\mathcal{L}(f(t))$					

Answer: First table left to right: $\frac{113t^2}{2}$,
$e^{-4t/3}$,
$-2\sin 4t$,
$\pi\cos 2t$,
$e^{-(t-2)}u(t-2)$.
The unit step $u(t)$ is defined in the Edwards-Penney textbook, $u(t) = 1$ for $t \geq 0$, zero elsewhere.
Second table left to right: $\frac{1}{s^2}$,
$\frac{1}{(s-1)^2} + \frac{1}{s+1}$,
$\left.\frac{s}{s^2+4}\right|_{s\to(s-1)} = \frac{s-1}{(s-1)^2+4}$,
$\left.\frac{6}{s^4}\right|_{s\to(s+1)} = \frac{6}{(s+1)^4}$,
$-\frac{d}{ds}\frac{s}{s^2+1} = \frac{s^2-1}{(s^2+1)^2}$

[30%] **Ch10(b)**: Compute $\mathcal{L}(f(t))$ for the pulse $f(t) = 100$ on $2 \leq t < 3$ and $f(t) = 0$ elsewhere.

Answer: Define $u(t)$ to be the unit step. Use $f(t) = 100(u(t-2) - u(t-3))$ and the second shifting theorem. Then $\mathcal{L}(100u(t-2)) = e^{-2s}\mathcal{L}(100|_{t\to t+2}) = e^{-2s}\mathcal{L}(100) = 100e^{-2s}/s$. Similarly, $\mathcal{L}(100u(t-3)) = 100e^{-3s}/s$. The answer: $\mathcal{L}(f(t)) = 100\left(e^{-2s} - e^{-3s}\right)/s$.

[30%] **Ch10(c)**: Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \frac{e^{-\pi s}}{s^2+9}$.

Answer: First write $\frac{1}{s^2+9} = \mathcal{L}(\frac{1}{3}\sin(3t))$ using the backward Laplace table. We use the second shifting theorem: $e^{-as}\mathcal{L}(h(t)) = \mathcal{L}(h(t-a)\,\mathbf{step}(t-a))$. Then $\mathcal{L}(f(t)) = e^{-\pi s}\mathcal{L}(\frac{1}{3}\sin(3t)) = \mathcal{L}(\frac{1}{3}\sin(3z)u(z)\Big|_{z=t-\pi}) = \frac{1}{3}\mathcal{L}(\sin(3t-3\pi)u(t-\pi))$. Lerch's theorem implies $f(t) = \frac{1}{3}\sin(3t-3\pi)u(t-\pi) = -\frac{1}{3}\sin(3t)u(t-\pi)$.

Place this page on top of all Ch10 work.