3. (Chapter 5) Complete all.

(3a) [50%] The differential equation \( \frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 12x^2 + 6x \) has a particular solution \( y_p(x) \) of the form \( y = d_1x^2 + d_2x^3 + d_3x^4 \). Find \( y_p(x) \) by the method of undetermined coefficients (yes, find \( d_1, d_2, d_3 \)).

Answer:

Solution \( y_h \) is a linear combination of the atoms \( 1, x, \cos(x), \sin(x) \). A particular solution is \( y_p = x^4 + x^3 - 12x^2 \).

The atoms for \( y'''' + y'' = 0 \) are found from \( r^4 + r^2 = 0 \) with roots \( r = 0, 0, i, -i \). The atoms in \( f(x) = 12x^2 + 6x \) are \( 1, x, x^2 \). Because \( 1, x \) are solutions of the homogeneous equation, then the list \( 1, x, x^2 \) from \( f(x) \) is multiplied by \( x^2 \) to obtain the corrected list \( x^2, x^3, x^4 \). Then \( y_p = d_1x^2 + d_2x^3 + d_3x^4 \).

Substitute \( y_p \) into the equation \( y'''' + y'' = 12x^2 + 6x \) to get \( 24d_3 + 2d_1 + 6d_2x + 12d_3x^2 = 12x^2 + 6x \). Matching coefficients of atoms gives \( 24d_3 + 2d_1 = 0, 6d_2 = 6, 12d_3 = 12 \). Then \( d_3 = 1, d_2 = 1, d_1 = -12 \). Finally, \( y_p = (-12)x^2 + (1)x^3 + (1)x^4 \).

(3b) [20%] Given \( 5x''(t) + 2x'(t) + 4x(t) = 0 \), which represents a damped spring-mass system with \( m = 5, c = 2, k = 4 \), determine if the equation is over-damped, critically damped, or under-damped.

To save time, do not solve for \( x(t) \)!

Answer:

Use the quadratic formula to decide. The number under the radical sign in the formula, called the discriminant, is \( b^2 - 4ac = 2^2 - 4(5)(4) = (19)(-4) \), therefore there are two complex conjugate roots and the equation is under-damped. Alternatively, factor \( 5r^2 + 2r + 4 \) to obtain roots \( (-1 \pm \sqrt{19}i)/5 \) and then classify as under-damped.

(3c) [30%] Given the forced spring-mass system \( x'' + 2x' + 17x = 82 \sin(5t) \), find the steady-state periodic solution.

Answer:

The answer is the undetermined coefficients solution \( x_p(t) = A \cos(5t) + B \sin(5t) \), because the homogeneous solution \( x_h(t) \) has limit zero at \( t = \infty \). Substitute the trial solution into the differential equation. Then \( -8A \cos(5t) - 8B \sin(5t) - 10A \sin(5t) + 10B \cos(5t) = 82 \sin(5t) \).

Matching coefficients of sine and cosine gives the equations \( -8A + 10B = 0, -10A - 8B = 82 \).

Solving, \( A = -5, B = -4 \). Then \( x_p(t) = -5 \cos(5t) - 4 \sin(5t) \) is the unique periodic steady-state solution.

Use this page to start your solution. Attach extra pages as needed.
4. (Chapter 5) Complete all.

(4a) [60%] A homogeneous linear differential equation with constant coefficients has characteristic equation of order 6 with roots 0, 0, −1, −1, 2i, −2i, listed according to multiplicity. The corresponding non-homogeneous equation for unknown $y(x)$ has right side $f(x) = 5e^{-x} + 4x^2 + x\cos 2x + \sin 2x$. Determine the undetermined coefficients shortest trial solution for $y_p$.

To save time, do not evaluate the undetermined coefficients and do not find $y_p(x)$! Undocumented detail or guessing earns no credit.

Answer:

The Euler solution atoms for roots of the characteristic equation are $1, x, e^{-x}, xe^{-x}, \cos 2x, \sin 2x$.

The atom list for $f(x)$ is $e^{-x}, 1, x, x^2, \cos 2x, x \cos 2x, \sin 2x, x \sin 2x$. This list of 8 atoms is broken into 4 groups, each group having exactly one base atom: (1) $1, x, x^2$, (2) $e^{-x}$, (3) $\cos 2x, x \cos 2x$, (4) $\sin 2x, x \sin 2x$. Each group contains a solution of the homogeneous equation. The modification rule is applied to groups 1 through 4. The trial solution is a linear combination of the replacement 8 atoms in the new list (1*) $x^2$, $x^3$, $x^4$, (2*) $x^2 e^{-x}$, (3*) $x \cos 2x, x^2 \cos 2x$, (4*) $x \sin 2x, x^2 \sin 2x$.

(4b) [40%] Let $f(x) = x^3e^{1.2x} + x^2e^{-x}\sin(x)$. Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation of least order which has $f(x)$ as a solution. To save time, do not expand the polynomial and do not find the differential equation.

Answer:

The characteristic polynomial is the expansion $(r - 1.2)^3((r + 1)^2 + 1)^3$. Because $x^3e^{ax}$ is an Euler solution atom for the differential equation if and only if $e^{ax}, xe^{ax}, x^2e^{ax}, x^3e^{ax}$ are Euler solution atoms, then the characteristic equation must have roots 1.2, 1.2, 1.2, 1.2, listing according to multiplicity. Similarly, $x^2e^{-x}\sin(x)$ is an Euler solution atom for the differential equation if and only if $-1 \pm i, -1 \pm i, -1 \pm i$ are roots of the characteristic equation. Total of 10 roots with product of the factors $(r - 1)^6((r + 1)^2 + 1)^3$ equal to the 10th degree characteristic polynomial.

Use this page to start your solution. Attach extra pages as needed.
5. (Chapter 6) Complete all parts.

(5a) True and False. No details required.
[10%] True or False (circle the answer)
The matrix $A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ fails to be a diagonalizable matrix.
[10%] True or False (circle the answer)
The matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ has only one eigenpair.
[10%] True or False (circle the answer)
If a $2 \times 2$ real matrix $A$ has a complex eigenpair $(2 + i, \vec{v}_1)$ with $\vec{v}_1 = \begin{pmatrix} i \\ -1 \end{pmatrix}$, then another eigenpair is $(2 - i, \vec{v}_2)$ where $\vec{v}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

Answer: False, True, True

(5b) [40%] Given $A = \begin{pmatrix} -2 & 2 & -1 & 2 \\ 0 & -2 & 5 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & 1 \end{pmatrix}$, which has eigenvalues $-2, -2, 1 + 3i, 1 - 3i$, display all solution details for finding all eigenvectors for eigenvalue $-2$.
To save time, do not find the eigenvectors for the other eigenvalues.

Answer:
One frame sequence is required for $\lambda = -2$. Subtract $-2$ from the diagonal of $A$ to obtain a homogeneous system of the form $B \vec{x} = \vec{0}$ (this is $(A-\lambda I) \vec{x} = \vec{0}$). The sequence starts with $\begin{pmatrix} 0 & 2 & -1 & 2 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & -3 & 3 \end{pmatrix}$,
the last frame having one row of zeros: $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. There is one invented symbol $t_1$ in the last frame algorithm answer $x_1 = t_1, x_2 = 0, x_3 = 0, x_4 = 0$. Taking $\partial t_1$ gives one eigenvector, $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

(5c) [20%] Find the matrices $P, D$ in the diagonalization equation $AP = PD$ for the matrix $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$.

Answer:
The eigenpairs of matrix $A$ are $\left( 1, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right), \left( 4, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right)$. Then $P = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

Use this page to start your solution. Attach extra pages as needed.