Name ____

Differential Equations and Linear Algebra 2250

Midterm Exam 3

Version 1a, 18apr2013

Scores
3.
4.
5.

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

3. **(Chapter 5)** Complete all.

(3a) [50%] The differential equation $\dfrac{d^4y}{dx^4} + \dfrac{d^2y}{dx^2} = 12x^2 + 6x$ has a particular solution $y_p(x)$ of the form $y = d_1x^2 + d_2x^3 + d_3x^4$. Find $y_p(x)$ by the method of undetermined coefficients (yes, find d_1, d_2, d_3).

Answer:

Solution y_h is a linear combination of the atoms $1, x, \cos(x), \sin(x)$. A particular solution is $y_p = x^4 + x^3 - 12x^2$.

The atoms for $y^(4) + y'' = 0$ are found from $r^4 + r^2 = 0$ with roots $r = 0, 0, i, -i$. The atoms in $f(x) = 12x^2 + 6x$ are $1, x, x^2$. Because $1, x$ are solutions of the homogeneous equation, then the list $1, x, x^2$ from $f(x)$ is multiplied by x^2 to obtain the corrected list x^2, x^3, x^4. Then $y_p = d_1x^2 + d_2x^3 + d_3x^4$.

Substitute y_p into the equation $y^(4) + y'' = 12x^2 + 6x$ to get $24d_3 + 2d_1 + 6d_2x + 12d_3x^2 = 12x^2 + 6x$. Matching coefficients of atoms gives $24d_3 + 2d_1 = 0, 6d_2 = 6, 12d_3 = 12$. Then $d_3 = 1, d_2 = 1, d_1 = -12$. Finally, $y_p = (-12)x^2 + (1)x^3 + (1)x^4$.

(3b) [20%] Given $5x''(t) + 2x'(t) + 4x(t) = 0$, which represents a damped spring-mass system with $m = 5$, $c = 2$, $k = 4$, determine if the equation is over-damped , critically damped or under-damped. **To save time, do not solve for $x(t)$!**

Answer:

Use the quadratic formula to decide. The number under the radical sign in the formula, called the discriminant, is $b^2 - 4ac = 2^2 - 4(5)(4) = (19)(-4)$, therefore there are two complex conjugate roots and the equation is **under-damped**. Alternatively, factor $5r^2 + 2r + 4$ to obtain roots $(-1 \pm \sqrt{19}i)/5$ and then classify as **under-damped**.

(3c) [30%] Given the forced spring-mass system $x'' + 2x' + 17x = 82\sin(5t)$, find the steady-state periodic solution.

Answer:

The answer is the undetermined coefficients solution $x_p(t) = A\cos(5t) + B\sin(5t)$, because the homogeneous solution $x_h(t)$ has limit zero at $t = \infty$. Substitute the trial solution into the differential equation. Then $-8A\cos(5t) - 8B\sin(5t) - 10A\sin(5t) + 10B\cos(5t) = 82\sin(5t)$. Matching coefficients of sine and cosine gives the equations $-8A + 10B = 0$, $-10A - 8B = 82$. Solving, $A = -5, B = -4$. Then $x_p(t) = -5\cos(5t) - 4\sin(5t)$ is the unique periodic steady-state solution.

Use this page to start your solution. Attach extra pages as needed.

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4. (Chapter 5) Complete all.

(4a) [60%] A homogeneous linear differential equation with constant coefficients has characteristic equation of order 6 with roots 0, 0, -1, -1, $2i$, $-2i$, listed according to multiplicity. The corresponding non-homogeneous equation for unknown $y(x)$ has right side $f(x) = 5e^{-x} + 4x^2 + x\cos 2x + \sin 2x$. Determine the undetermined coefficients **shortest** trial solution for y_p.
To save time, do not evaluate the undetermined coefficients and do not find $y_p(x)$! Undocumented detail or guessing earns no credit.

Answer:

The Euler solution atoms for roots of the characteristic equation are $1, x, e^{-x}, xe^{-x}, \cos 2x, \sin 2x,$. The atom list for $f(x)$ is e^{-x}, $1, x, x^2$, $\cos 2x$, $x\cos 2x$, $\sin 2x$, $x\sin 2x$. This list of 8 atoms is broken into 4 groups, each group having exactly one base atom: (1) $1, x, x^2$, (2) e^{-x}, (3) $\cos 2x$, $x\cos 2x$, (4) $\sin 2x$, $x\sin 2x$. Each group contains a solution of the homogeneous equation. The modification rule is applied to groups 1 through 4. The trial solution is a linear combination of the replacement 8 atoms in the new list (1*) x^2, x^3, x^4, (2*) x^2e^{-x}, (3*) $x\cos 2x$, $x^2\cos 2x$ (4*) $x\sin 2x$, $x^2\sin 2x$.

(4b) [40%] Let $f(x) = x^3e^{1.2x} + x^2e^{-x}\sin(x)$. Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation of least order which has $f(x)$ as a solution. To save time, do not expand the polynomial and do not find the differential equation.

Answer:

The characteristic polynomial is the expansion $(r-1.2)^4((r+1)^2+1)^3$. Because x^3e^{ax} is an Euler solution atom for the differential equation if and only if $e^{ax}, xe^{ax}, x^2e^{ax}, x^3e^{ax}$ are Euler solution atoms, then the characteristic equation must have roots $1.2, 1.2, 1.2, 1.2$, listing according to multiplicity. Similarly, $x^2e^{-x}\sin(x)$ is an Euler solution atom for the differential equation if and only if $-1 \pm i, -1 \pm i, -1 \pm i$ are roots of the characteristic equation. Total of 10 roots with product of the factors $(r-1)^4((r+1)^2+1)^3$ equal to the 10th degree characteristic polynomial.

Use this page to start your solution. Attach extra pages as needed.

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5. (Chapter 6) Complete all parts.

(5a) True and False. No details required.
[10%] True or False (circle the answer)
The matrix $A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ fails to be a diagonalizable matrix.
[10%] True or False (circle the answer)
The matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ has only one eigenpair.
[10%] True or False (circle the answer)
If a 2×2 real matrix A has a complex eigenpair $(2+i, \vec{v}_1)$ with $\vec{v}_1 = \begin{pmatrix} i \\ -1 \end{pmatrix}$, then another eigenpair is $(2-i, \vec{v}_2)$ where $\vec{v}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

Answer:

False, True, True

(5b) [40%] Given $A = \begin{pmatrix} -2 & 2 & -1 & 2 \\ 0 & -2 & 5 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & 1 \end{pmatrix}$, which has eigenvalues $-2, -2, 1+3i, 1-3i$, display all solution details for finding **all** eigenvectors for eigenvalue -2.
To save time, do not find the eigenvectors for the other eigenvalues.

Answer:

One frame sequence is required for $\lambda = -2$. Subtract -2 from the diagonal of A to obtain a homogeneous system of the form $B\vec{x} = \vec{0}$ (this is $(A-\lambda I)\vec{x} = \vec{0}$). The sequence starts with $\begin{pmatrix} 0 & 2 & -1 & 2 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & -3 & 3 \end{pmatrix}$, the last frame having one row of zeros: $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. There is one invented symbol t_1 in the last frame algorithm answer $x_1 = t_1$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$. Taking ∂_{t_1} gives one eigenvector, $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

(5c) [20%] Find the matrices P, D in the diagonalization equation $AP = PD$ for the matrix $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$.

Answer:

The eigenpairs of matrix A are $\left(1, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$, $\left(4, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$. Then $P = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

Use this page to start your solution. Attach extra pages as needed.