Name ______________________________________

Differential Equations and Linear Algebra 2250

Midterm Exam 3
Version 1, Thu 11 April 2013

Scores
1.
2.

Instructions: This in-class exam is designed for 25 minutes. Exam period 7:30 to 9:40. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Chapter 10)** Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(1a) [40%] Display the details of Laplace's method to solve the system for $x(t)$. Don't solve for $y(t)$!

$$\begin{aligned} x' &= x + 3y, \\ y' &= -2y, \\ x(0) &= 1, \quad y(0) = 2. \end{aligned}$$

Lectures: *Effort can be saved by using the Laplace resolvent shortcut equation* $(sI - A)\mathcal{L}(\mathbf{u}) = \mathbf{u}(0)$ *and Cramer's Rule. Notation:* $\mathbf{u}$ *is the vector solution of* $\mathbf{u}' = A\mathbf{u}$ *with components* $x(t)$, $y(t)$. *The same credit applies if you don't use this shortcut, but solve the problem correctly using Laplace theory.*

Answer:

The Laplace resolvent equation $(sI - A)\mathcal{L}(\mathbf{u}) = \mathbf{u}(0)$ can be written out to find a 2×2 linear system for unknowns $\mathcal{L}(x(t))$, $\mathcal{L}(y(t))$:

$$(s-1)\mathcal{L}(x) + (-3)\mathcal{L}(y) = 1, \quad (0)\mathcal{L}(x) + (s+2)\mathcal{L}(y) = 2.$$

Any other method of arriving at the system of linear algebraic equations is acceptable, it is not an error to do it another way. Elimination or Cramer's rule or matrix inversion applies to this system to solve for $\mathcal{L}(x(t)) = \dfrac{(s+2)+6}{(s-1)(s+2)} = \dfrac{-2}{s+2} + \dfrac{3}{s-1}$. Then the backward table implies $x(t) = -2e^{-2t} + 3e^{t}$.

(1b) [30%] Find $f(t)$ by partial fractions, the shifting theorem and the backward table, given

$$\mathcal{L}(f(t)) = \frac{2s^3 + 3s^2 - 6s + 3}{s^3(s-1)^2}.$$

Answer:

The numerator has degree 3, less than the denominator degree 5. Partial fraction theory gives an expansion $\frac{2s^3+3s^2-6s+3}{s^3(s-1)^2} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s^3} + \frac{d}{s-1} + \frac{e}{(s-1)^2}$. We find the system of equations and solve it for $a = b = 0$, $c = 3$, $d = 0$, $e = 2$. Then $\mathcal{L}(f(t)) = \frac{3}{s^3} + \frac{2}{(s-1)^2} = \mathcal{L}(3t^2/2 + 2te^t)$ implies $f(t) = 3t^2/2 + 2te^t$.

(1c) [30%] Solve for $f(t)$, given

$$\mathcal{L}(e^{2t}f(t)) + 2\frac{d^2}{ds^2}\mathcal{L}(tf(t)) = \frac{s+3}{(s+1)^3}.$$

Answer:

Use the s-differentiation theorem, partial fractions and the backward Laplace table plus the shift theorem to get $\mathcal{L}(e^{2t}f(t)) + 2\mathcal{L}((-t)^2(t)f(t)) = \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3} = \mathcal{L}(te^{-t} + t^2e^{-t})$. Lerch's theorem implies $(e^{2t} + 2t^3)f(t) = te^{-t} + (t^2)e^{-t}$. Then $f(t) = (t + t^2)e^{-t}/(e^{2t} + 2t^3)$.

Use this page to start your solution. Attach extra pages as needed.

Name. ______________________________

2. (Chapter 10) Complete all parts.

(2a) [60%] Fill in the blank spaces in the Laplace table:

Forward Table

$f(t)$	$\mathcal{L}(f(t))$
t^3	$\dfrac{6}{s^4}$
$e^{-t}\cos(4t)$	
$(t+2)^2$	
t^2e^{-2t}	

Backward Table

$\mathcal{L}(f(t))$	$f(t)$
$\dfrac{3}{s^2+9}$	$\sin 3t$
$\dfrac{s-1}{s^2-2s+5}$	
$\dfrac{2}{(2s-1)^2}$	
$\dfrac{s}{(s-1)^3}$	

Answer:

Forward: $\dfrac{s+1}{(s+1)^2+16}$, $\dfrac{2}{s^3}+\dfrac{4}{s^2}+\dfrac{4}{s}$, $\frac{2}{(s+2)^3}$.

Backward: $e^t\cos(2t)$, $(1/2)te^{t/2}$, $te^t+(t^2/2)e^t$.

(2b) [20%]

Find $\mathcal{L}(f(t))$ from the Second Shifting theorem, given $f(t)=\sin(2t)\mathbf{u}(t-2)$, where $\mathbf{u}$ is the unit step function defined by $\mathbf{u}(t)=1$ for $t\ge 0$, $\mathbf{u}(t)=0$ for $t<0$.

Answer:

Use the second shifting theorem

$$\mathcal{L}(g(t)u(t-a)) = e^{-as}\mathcal{L}\left(g(t)|_{t\to t+a}\right).$$

Write $g(t)=\sin(2t)$.

Then $\mathcal{L}(f(t))=\mathcal{L}(g(t)u(t-2))=e^{-2s}\mathcal{L}(g(t)|_{t->t+2})=e^{-2s}\mathcal{L}(\sin(2t+4))=e^{-2s}\mathcal{L}(\sin(2t)\cos(4)+\sin(4)\cos(2t))$. Then Lerch's law implies $\mathcal{L}(f(t))=e^{-2s}\left(\frac{2\cos 4}{s^2+4}+\frac{s\sin(4)}{s^2+4}\right)$.

(2c) [20%] Find $f(t)$ from the Second Shifting Theorem, given $\mathcal{L}(f(t))=\dfrac{s\,e^{-\pi s}}{s^2+2s+17}$.

Answer:

Use the second shifting theorem in backward form

$$e^{-as}\mathcal{L}(f(t)) = \mathcal{L}(f(t-a)\mathbf{u}(t-a)).$$

Then $\mathcal{L}(f(t))=\dfrac{s\,e^{-\pi s}}{(s^2+2s+1)+16}=e^{-\pi s}\mathcal{L}(e^{-t}\cos(4t))=\mathcal{L}(e^{-t}\cos(t)u(t)\Big|_{t\to(t-\pi)})=\mathcal{L}(e^{-t+\pi}\cos(t-\pi)u(t-\pi))$. Then Lerch's theorem implies $f(t)=e^{-t+\pi}(\cos(t)\cos(\pi)+\sin(t)\sin(\pi))\,u(t-\pi)$, which reduces to $f(t)=-e^{-t+\pi}\cos(t)u(t-\pi)$.

Use this page to start your solution. Attach extra pages as needed.