Differential Equations and Linear Algebra 2250
Midterm Exam 2
Version 1, 21 Mar 2013

Instructions: This in-class exam is designed to be completed in 30 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (The 3 Possibilities with Symbols)
Let $a$, $b$, and $c$ denote constants and consider the system of equations

$$
\begin{pmatrix}
0 & 0 & 0 \\
-2b - 4 & 3 & a \\
b + 1 & -1 & 0 \\
-1 - b & 1 & a
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} =
\begin{pmatrix}
0 \\
b^2 \\
b \\
b^2 - b
\end{pmatrix}
$$

(a) [40%] Determine $a$ and $b$ such that the system has a unique solution.

(b) [30%] Explain why $a = 0$ and $b \neq 0$ implies no solution. Ignore any other possible no solution cases.

(c) [30%] Explain why $a = b = 0$ implies infinitely many solutions. Ignore any other possible infinitely many solution cases.

Because of a row of zeros (equation $0 = 0$), we can reduce the question to a 3x3 problem $A\mathbf{u} = \mathbf{b}$ when

$$
A = \begin{pmatrix}
-2b - 4 & 3 & a \\
b + 1 & -1 & 0 \\
-1 - b & 1 & a
\end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b^2 \\ b \\ b^2 - b \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}
$$

Then $|A| = a - ab = a(1-b) \neq 0$ for $a \neq 0$ and $b \neq 1$

(a) Unique sol when $|A| \neq 0$ (see reduction above)

\[ a \neq 0 \text{ and } b \neq 1 \]

(b) when $a = 0$, then $A = \begin{pmatrix} -2b - 4 & 3 & 0 \\ b + 1 & -1 & 0 \\ -1 - b & 1 & 0 \end{pmatrix}$ has col rank $\leq 2$

hence there is one free var, if the system is consistent. We do

ref steps on $C = \langle A \mid B \rangle$ to get

$$
\begin{pmatrix}
2b - 4 & 3 & 0 \\
b + 1 & -1 & 0 \\
0 & 0 & b^2
\end{pmatrix}
$$

hence no solution for $b \neq 0$.

(c) Define $C$ as in part (b), then substitute $a = b = 0$ to get

ref step $\begin{pmatrix} -4 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. A homogeneous system is consistent.

Use this page to start your solution. Attach extra pages as needed.
2. (Vector Spaces) Do all parts. Details not required for (a)-(d).

(a) [10%] True or false: There is a subspace $S$ of $\mathbb{R}^3$ containing none of the vectors $(1, 1, 1), (1, -1, 1), (3, 0, 2)$. 

(b) [10%] True or false: The set of solutions $\vec{u}$ in $\mathbb{R}^3$ of a consistent matrix equation $A\vec{u} = \vec{b}$ can equal all vectors in the $xy$-plane, that is, all vectors of the form $\vec{u} = (x, y, 0)$.

(c) [10%] True or false: Relations $x^2 + y^2 = 0, 0 + z = 0$ define a subspace in $\mathbb{R}^3$.

(d) [10%] True or false: Equations $x = y, z = 2y$ define a subspace in $\mathbb{R}^3$.

(e) [20%] Linear algebra theorems are able to conclude that the set $S$ of all polynomials $f(x) = c_0 + c_1x + c_2x^2$ such that $f'(x) + \int_0^1 f(x) dx = 0$ is a vector space of functions. Explain why $V = \text{span}(1, x, x^2)$ is a vector space, then fully state a linear algebra theorem required to show $S$ is a subspace of $V$. To save time, do not write any subspace proof details.

(f) [40%] Find a basis of vectors for the subspace of $\mathbb{R}^3$ given by the system of restriction equations

\[
\begin{align*}
3x_1 + 2x_3 + 4x_4 + 10x_5 &= 0, \\
2x_1 + x_3 + 2x_4 + 4x_5 &= 0, \\
-2x_1 + 4x_5 &= 0, \\
2x_1 + 2x_3 + 4x_4 + 12x_5 &= 0.
\end{align*}
\]

(c) $x^2 + y^2 = 0 \iff \{x = 0, y = 0\}$ which are 2 linear algebraic eqs.

Rest follows from The Kernel Theorem.

(e) $V = \text{span}(1, x, x^2) = \text{subspace of } \mathbb{R}^3$ vector space of all polynomials, by the Span Theorem (Sec. 4.2)

$S$ is a subspace by the Subspace Criterion (Section 4.1)

Theorem $S$ is a subspace of vector space $V$ provided

1. $0$ is in $S$; 2. $x, y$ in $S \Rightarrow x + y$ in $S$; 3. $x$ in $S$, $c = \text{const} \Rightarrow cx$ in $S$

4. Variable $x_2$ missing, so its a free variable. Augmented matrix is

\[
C = \begin{pmatrix}
3 & 0 & 2 & 4 & 10 & 0 \\
2 & 0 & 1 & 2 & 4 & 0 \\
-2 & 0 & 0 & 4 & 12 & 0
\end{pmatrix}, \quad \text{ref}(C) = \begin{pmatrix}
1 & 0 & 0 & -2 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Use this page to start your solution. Attach extra pages as needed.
3. (Independence and Dependence) Do all parts.
   (a) [10%] State a dependence test for 3 vectors in \( \mathbb{R}^4 \). Write the hypothesis and conclusion, not just the name of the test.
   (b) [10%] State fully an independence test for 3 polynomials. It should apply to show that \( 1, 1 + x, x(1 + x) \) are independent.
   (c) [10%] For any matrix \( A \), \( \text{rank}(A) \) equals the number of lead variables for the problem \( A\vec{x} = \vec{0} \). How many non-pivot columns in an \( 8 \times 8 \) matrix \( A \) with \( \text{rank}(A) = 6 \)?
   (d) [30%] Let \( v_1, v_2, v_3, v_4 \) denote the rows of the matrix

   \[
   A = \begin{pmatrix}
   0 & -2 & 0 & -6 & 0 \\
   0 & 2 & 0 & 5 & 1 \\
   0 & 1 & 0 & 2 & 1 \\
   0 & 1 & 0 & 3 & 0
   \end{pmatrix}
   \]

   Decide if the four rows \( \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \) are independent and display the details of the chosen independence test.
   (e) [40%] Extract from the list below a largest set of independent vectors.

   \[
   \vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} 0 \\ 3 \\ -1 \\ 5 \end{pmatrix}, \quad \vec{v}_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \quad \vec{v}_6 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}
   \]

   (A) Rank Test: 3 vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) are dependent \( \iff \) the rank of the augmented matrix has rank \( \leq 3 \).

   (B) Wronskian Test: If the Wronskian of the 3 polynomials is nonzero at some \( x = x_0 \), then the 3 polynomials are independent.

   (C) The rank = # pivot cols, therefore \( \text{ans} = 2 \).

   (D) The col rank is \( \leq 3 \), so the 4 rows cannot be independent, by the rank test applied to the transpose of \( A \).

   (E) The safest method is to form the augmented matrix \( A \) of the col vectors (\( A \) is \( 6 \times 6 \)). Then find the pivot cols from \( \text{ref}(A) \). The answer is \( \begin{pmatrix} \vec{v}_2 & \vec{v}_4 \end{pmatrix} \). It is also possible to see \( \vec{v}_3 = \frac{1}{2} \vec{v}_2 \). \( \vec{v}_5 = \vec{v}_4 - \vec{v}_2 \). \( \vec{v}_6 = \vec{v}_5 - \vec{v}_3 \). So \( \vec{v}_2, \vec{v}_4 \) are the independent columns.

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4. (Determinants) Do all parts.
   (a) [10%] True or False: The value of a determinant is the product of the diagonal elements.
   (b) [10%] True or False: The determinant of the negative of the $n \times n$ identity matrix is $-1$.
   (c) [20%] Assume given $3 \times 3$ matrices $A, B$. Suppose $A^2B = E_2E_1A$ and $E_1, E_2$ are elementary matrices representing respectively a swap and a multiply by $-5$. Assume $\det(B) = 10$. Let $C = 2A$. Find all possible values of $\det(C)$.
   (d) [30%] Determine all values of $x$ for which $(I + C)^{-1}$ fails to exist, where $C$ equals the transpose of the matrix \[
\begin{pmatrix}
2 & 0 & -1 \\
3x & 0 & 1 \\
x-1 & x & x
\end{pmatrix}
\]
   (e) [30%] Let symbols $a, b, c$ denote constants. Apply the adjugate [adjoint] formula for the inverse to find the value of the entry in row 3, column 4 of $A^{-1}$, given $A$ below.

\[
A = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
a & b & 0 & 1 \\
1 & c & 1 & 2
\end{pmatrix}
\]

Use this page to start your solution. Attach extra pages as needed.
5. (Linear Differential Equations) Do all parts.
(a) [20%] Solve for the general solution of $15y'' + 8y' + y = 0$.
(b) [40%] The characteristic equation is $r^2(2r + 1)^3(r^2 - 2r + 10) = 0$. Find the general solution $y$ of the linear homogeneous constant-coefficient differential equation.
(c) [20%] A fourth order linear homogeneous differential equation with constant coefficients has two particular solutions $2e^{3x} + 4x$ and $xe^{3x}$. Write a formula for the general solution.
(d) [20%] Mark with $X$ the functions which cannot be a solution of a linear homogeneous differential equation with constant coefficients. Test your choices against this theorem:

The general solution of a linear homogeneous $n$th order differential equation with constant coefficients is a linear combination of Euler solution atoms.

| $X$ | $e^{ln(2x)}$ | $X$ | $e^{x^2}$ | $2x + x$ | $cos(ln|x|)$ |
|-----|--------------|-----|----------|---------|-------------|
| $c_1$ | $e^{-x^2}$ | $c_2$ | $e^{x^2}$ | $c_3$ | $c_4$ |
| $cos(x ln(3.7125))$ | $x^{-1}e^{-x}sin(\pi x)$ | $cosh(x)$ | $sin^2(x)$ |

(a) $15r^2 + 8r + 1 = 0 \Rightarrow (5r + 1)(3r + 1) = 0 \Rightarrow r = -\frac{1}{5}, -\frac{1}{3}$

$y = c_1 e^{-x/5} + c_2 e^{-x/3}$

(b) roots: $r = 0, 0, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 1 \pm 3i$

Euler Sol atoms = $1, x, e^{-x/2}, x e^{-x/2}, x^2 e^{-x/2}, e^x \cos 3x, e^x \sin 3x$

$y = \text{linear combination of Euler atoms}$

(c) Atoms must be $1, x, e^{3x}, x e^{3x}$ and the general solution is a linear combination of those 4 Euler Sol. atoms.

(d) $e^{ln(2x)} =12x + 1$ not an atom or l.c. of atoms

$cos(ln(x))$ not an atom, only $cos(ln(x))$ qualifies

$x^{-1}$ not allowed, only positive powers or zero power

$e^{-x^2}$ not of the form $e^{ax}$.

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