Linear Algebra 2270-2
Due in Week 8

The eighth week starts the work from chapter 4. Here's the list of problems, followed by problem notes and a few answers.

Section 4.1. Exercises 5, 9, 11, 12, 16, 17, 19, 20, 21, 26

Section 4.2. Exercises 1, 2, 3, 11, 12, 17, 21, 27, 31

Section 4.3. Exercises 1, 6, 12, 17, 18, 21

Problem Notes

Issues for Strang's problems will be communicated here. For help, please send email, call 581-6879, or visit JWB 113.

Some Answers

4.1. Exercises 9, 12, 16, 21 have textbook answers.

4.1-5. The problem is about the equation nullspace = rowspace$^\perp$, valid for both matrix A and A^T. The **Fundamental Theorem of Linear Algebra**, Part II, can be summarized as the text

> *The nullspace is perpendicular to the rowspace.*

The text is justified from equality of vectors: $Ax = 0$ is equivalent to scalar equations $\mathbf{row}(A,1)\cdot x = 0, \ldots, \mathbf{row}(A,m)\cdot x = 0$, which says that x is perpendicular to all rows of A, hence perpendicular to the rowspace of A.

(a) Apply the equation to A^T. Then nullspace(A^T) $\perp$ rowspace(A^T), equivalent to nullspace(A^T) $\perp$ colspace(A), implies any solution y to $A^Ty = 0$ is perpendicular to any Ax. Since $b = Ax$, then $y \perp b$ or $y^Tb = 0$.
(b) If $A^Ty = (1,1,1)$ has a solution, then y is in rowspace(A). Then nullspace(A) $\perp$ rowspace(A) implies $y \cdot x = 0$ for all x in the nullspace of A.

4.1-9. Answers: colspace, perpendicular. See problem 4.1-5 above for nullspace $\perp$ rowspace, applied here for matrix A^T.
Prove $A^TAx = 0$ implies $Ax = 0$.
Because $y = Ax$ is a linear combination of the columns of A, then y is in colspace(A) = rowspace(A^T). If $A^TAx = 0$, then $A^Ty = 0$, which implies y is in nullspace(A^T). Use nullspace(A^T) $\perp$ rowspace(A^T). Then nullspace(A) and rowspace(A^T) meet only in the vector $y = 0$, which says $y = Ax = 0$.
Prove $Ax = 0$ implies $A^TAx = 0$.
First, assume $Ax = 0$. Multiply by A^T to get $A^TAx = A^T0$. The right side is the zero vector, which gives $A^TAx = 0$.
See also problem 4.2-27, which repeats this same argument.

4.1-11.

For A: The nullspace is spanned by $(-2,1)$, the row space is spanned by $(1,2)$. The column space is the line through $(1,3)$ and $N(A^T)$ is the line through $(3,-1)$. In each case,

For B: The nullspace of B is is a line spanned by $(0,1)$, the row space is a line spanned by $(1,0)$. The column space and left nullspace are the same as for A. As in (a), the line pairs are perpendicular.

4.1-17. If S is the subspace of $\mathcal{R}^3$ containing only the zero vector, then $S^\perp$ is $\mathcal{R}^3$. If S is spanned by $(1,1,1)$, then $S^\perp$ is the plane spanned by any two independent vectors perpendicular to $(1,1,1)$. For example, the vectors $(1,-1,0)$ and $(1,0,-1)$. If S is spanned by $(2,0,0)$ and $(0,0,3)$, then $S^\perp$ is the line spanned by $(0,1,0)$, computed as the cross product of the two vectors, then scaled to be a unit vector.

4.1-26. $A = \begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}$

This example shows a matrix with perpendicular columns. Then $A^TA = 9I$ is diagonal: $(A^TA)_{ij} =$ (column i of A) $\cdot$ (column j of A). When the columns are unit vectors, then $A^TA = I$.

4.2. Exercises 1, 3, 11, 21, 31 have textbook answers.

4.2-2.

(a) The projection of $b = (\cos\theta, \sin\theta)$ onto $a = (1,0)$ is $p = (\cos\theta, 0)$.

(b) The projection of $b = (1,1)$ onto $a = (1,-1)$ is $p = (0,0)$ since $a^Tb = 0$.

4.2-12. $P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ = projection matrix onto the column space of A (the xy plane)

$P_2 = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0.0 & 0.0 & 1 \end{pmatrix}$ = Projection matrix onto the second column space. Certainly $(P_2)^2 = P_2$.

4.2-17. If $P^2 = P$ then $(I-P)^2 = (I-P)(I-P) = I^2 - PI - IP + P^2 = I - P$. When P projects onto the column space, then $I - P$ projects onto the left nullspace.

4.2-27. If $A^TAx = 0$ then Ax is a vector in the nullspace of A^T. But Ax is a vector in the column space of A. To be in both of those perpendicular spaces, Ax must be zero. So A and A^TA have the same nullspace.

4.3. Exercises 1, 18, 21 have textbook answers.

4.3-6. $a = (1,1,1,1)$ and $b = (0,8,8,20)$ give $\hat{x} = \dfrac{a^Tb}{a^Ta} = 9$ and the projection is $\hat{x}a = p = (9,9,9,9)$. Then $e^Ta = (-9,-1,-1,11)^T(1,1,1,1) = 0$ and $\|e\| = \sqrt{204}$.

4.3-12.

(a) $a = (1,\ldots,1)$ has $a^Ta = m$, $a^Tb = b_1 + \cdots + b_m$. Therefore $\hat{x} = a^Tb/m$ is the mean of the b's

(b) $e = b - \hat{x}a$, $b = (1,2,b)$, $\|e\| = \sum_{i=1}^m (b_1 - \hat{x})^2$ = variance

(c) $p = (3,3,3)$, $e = (-2,-1,3)$, $p^Te = 0$. $P = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

4.3-17. $\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 21 \end{pmatrix}$. The solution $\vec{x} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$ comes from $\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 35 \\ 42 \end{pmatrix}$.