

Solutions

1. (5 points) Give a basis for each of the four fundamental subspaces associated to the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution:

The two row vectors form a basis for the row space.

The first two columns give a basis for the column space (but not all three columns, the second and third are not independent).

The left nullspace is the orthogonal complement in \mathbb{R}^2 of the column space. The column space is already all of \mathbb{R}^2 , so the left nullspace is trivial.

The nullspace is the orthogonal complement of the row space in \mathbb{R}^3 , so it is one dimensional. To find a vector in the nullspace, observe that columns 1 and 2 are pivot columns, so column 3 is a free column. Thus, there is a null vector of the form $\bar{v} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$. Solve $A\bar{v} = \bar{0}$ to find

$$\text{that } \bar{v} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

2. (5 points) V is the span of the given vectors in \mathbb{R}^4 . Find orthonormal vectors whose span is V .

$$\bar{v}_1 = \begin{pmatrix} 3 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \bar{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Solution:

\bar{v}_1 is not length 1, it has length $\sqrt{3^2 + 4^2} = 5$. Let \bar{b}_1 be the unit vector $\frac{\bar{v}_1}{5}$.

$$\bar{b}_1 \cdot \bar{v}_2 = \frac{1}{5}(3+4) = \frac{7}{5} \neq 0,$$

so \bar{v}_2 is not orthogonal to \bar{b}_1 . Subtract off the “ \bar{b}_1 -part” of \bar{v}_2 :

$$\bar{v}_2 - (\bar{v}_2 \cdot \bar{b}_1)\bar{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{7}{25} \begin{pmatrix} 3 \\ 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{25} \\ 1 \\ -\frac{3}{25} \\ 0 \end{pmatrix}$$

This is orthogonal to \bar{b}_1 , but has length

$$\sqrt{\left(\frac{4}{25}\right)^2 + 1^2 + \left(-\frac{3}{25}\right)^2} = \sqrt{1 + \frac{1}{25}} = \frac{1}{5}\sqrt{26}$$

$$\text{Let } \bar{b}_2 = \frac{1}{\frac{1}{5}\sqrt{26}} \begin{pmatrix} \frac{4}{25} \\ 1 \\ -\frac{3}{25} \\ 0 \end{pmatrix}$$

$\{\bar{b}_1, \bar{b}_2\}$ is an orthonormal basis of V .

3. (5 points) For the subspace V in the previous problem, give the matrix that projects \mathbb{R}^4 to V and the matrix that projects \mathbb{R}^4 to V^\perp .

Solution:

Let A be the matrix whose columns are the vectors \bar{b}_1 and \bar{b}_2 from the previous problem. Projection to V is given by the matrix $P_V = A(A^T A)^{-1}A^T$. Since A is orthogonal, $(A^T A)$ is the identity matrix, so

$$\begin{aligned} P_V &= AA^T = \frac{1}{25 \cdot 26} \begin{pmatrix} 3\sqrt{26} & 4 \\ 0 & 25 \\ 4\sqrt{26} & -3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3\sqrt{26} & 0 & 4\sqrt{26} & 0 \\ 4 & 5 & -3 & 0 \end{pmatrix} \\ &= \frac{1}{25 \cdot 26} \begin{pmatrix} 10 \cdot 25 & 20 & 12 \cdot 25 & 0 \\ 100 & 125 & -75 & 0 \\ 12 \cdot 25 & -15 & 17 \cdot 25 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{26} \begin{pmatrix} 10 & \frac{4}{5} & 12 & 0 \\ 4 & 5 & -3 & 0 \\ 12 & -\frac{3}{5} & 17 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

The projection to V^\perp is given by $ID - P_V$.

4. (5 points) Find the least squares best fit line for the points $(0, 1)$, $(2, 3)$, $(4, 4)$.

Solution:

Let

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix}$$

A line through the three points would have slope m and y -intercept b that give a solution $\bar{v} = \begin{pmatrix} m \\ b \end{pmatrix}$ to the equation

$$A\bar{v} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

However, $\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ is not in the column space of A , so there is no actual solution. The best fit line has slope and y -intercept given by

$$\bar{v} = (A^T A)^{-1} A^T \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 20 & 6 \\ 6 & 3 \end{pmatrix}$$

so

$$(A^T A)^{-1} = \frac{1}{24} \begin{pmatrix} 3 & -6 \\ -6 & 20 \end{pmatrix}$$

$$(A^T A)^{-1} A^T = \frac{1}{24} \begin{pmatrix} -6 & 0 & 6 \\ 20 & 8 & -4 \end{pmatrix}$$

$$\bar{v} = (A^T A)^{-1} A^T \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \frac{1}{24} \begin{pmatrix} 18 \\ 28 \end{pmatrix}$$

The best fit line is

$$y = \frac{18}{24}x + \frac{28}{24}$$

5. (5 points) Find the determinant of the following matrix:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution:

The determinant is 1. There are many ways to compute this.

6. (5 points) For the following matrix, find the eigenvalues and the maximum number of linearly independent eigenvectors. Find this many linearly independent eigenvectors.

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$$

Solution:

The characteristic polynomial is

$$\begin{vmatrix} 1 - \lambda & 3 \\ 2 & 2 - \lambda \end{vmatrix} = (1 - \lambda)(2 - \lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$$

The eigenvalues are the roots of the characteristic polynomial: 4 and -1. Since there are two distinct eigenvalues, there are at least two linearly independent eigenvectors, one for each eigenvalue. Since the space is two dimensional there can be at most two linearly independent eigenvectors.

To find an eigenvector \bar{v}_1 for $\lambda_1 = 4$ find the nullspace of $A - 4ID = \begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix}$. The second column is -1 times the first, so $\bar{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ works.

To find an eigenvector \bar{v}_2 for $\lambda_2 = -1$ find the nullspace of $A - (-1)ID = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$. The second column is $\frac{3}{2}$ times the first, so $\bar{v}_2 = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$ works.

7. (5 points) For the following matrix, find the eigenvalues and the maximum number of

linearly independent eigenvectors. Find this many linearly independent eigenvectors.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Solution:

The matrix is in Jordan Form. The eigenvalues are the diagonal entries 1, 2, 3. There are four Jordan blocks, so there are four linearly independent eigenvectors. For example:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

8. (5 points) Describe the plane in \mathbb{R}^3 that contains the three points $(1, 0, 0)$, $(1, 1, 1)$, $(1, 2, 0)$.

Solution:

The vector from $(1, 0, 0)$ to $(1, 1, 1)$ is $\bar{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. The vector from $(1, 0, 0)$ to $(1, 2, 0)$ is

$\bar{w} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$. The plane containing the three points is collection (not a vector subspace) of

vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$, where a and b are arbitrary real numbers.

In other words, it is the vector subspace spanned by \bar{v} and \bar{w} shifted by the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

9. (5 points) Suppose an $n \times n$ matrix A has all eigenvalues equal to 0. Show that A^n has all entries equal to 0.

Solution:

Solution 1: If all eigenvalues are 0 then the characteristic polynomial is λ^n . By the Cayley-Hamilton Theorem, A is a root of its characteristic polynomial, so A^n is the zero matrix.

Solution 2: A is similar to a matrix J in Jordan Form with 0's on the diagonal and possibly some 1's on the first super-diagonal (immediately above the diagonal). Check that J^k is a matrix that is zero everywhere except possibly for some 1's on the k -th super-diagonal, so J^n is the zero matrix. $A = CJC^{-1}$ so $A^n = CJ^nC^{-1}$ is also the zero matrix.

10. (1000000 points) Prove the Cayley-Hamilton Theorem for matrices with real eigenvalues. You may assume the Jordan Form Theorem.

Solution:

No new questions beyond this point.