MATH 2270-003 | Sample Exam 1 | Fall 2010

Solutions

1. (5 points) What is the magnitude of the vector $\begin{pmatrix}1\\2\end{pmatrix}$?

Solution:

$$\left\|\begin{pmatrix}1\\2\end{pmatrix}\right\| = \sqrt{\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix}} = \sqrt{5}$$

2. (5 points) What is the unit vector that has the same direction as $\begin{pmatrix}1\\2\end{pmatrix}$?

Solution:

$$\frac{\begin{pmatrix}1\\2\end{pmatrix}}{\left\|\begin{pmatrix}1\\2\end{pmatrix}\right\|} = \begin{pmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{pmatrix}$$

3. (5 points) What is the coefficient matrix corresponding to the following system of linear equations?

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= b_1 \\ 4x_1 + 2x_2 + 3x_3 &= b_2 \\ 4x_3 + 5x_4 &= b_3 \end{aligned}$$

Solution:

$$\begin{pmatrix}1 & 1 & 1 & 1\\ 4 & 2 & 3 & 0\\ 0 & 0 & 4 & 5\end{pmatrix}$$

4. (5 points) Let A be a 3×4 matrix. What is the elimination matrix that replaces Row 2 of A with $Row2 - Row1$ and replaces $Row3$ of A by $Row3 - 2Row1$?

Solution:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

5. (5 points) What is the inverse of the following matrix?

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Solution:

E is an elimination matrix. It replaces Row 4 with Row 4- 5 Row 3 and replaces Row 5 with Row 3 + Row 5. The inverse is the elimination matrix that reverses these steps, replacing Row 4 with Row 4 + 5Row 3 and replacing Row 5 with Row 5 - Row 3.

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

6. (5 points) Descibe in word the effect of multiplying EA where E is the matrix in the previous problem and A is a 5×5 matrix.

Solution:

E is an elimination matrix. It replaces Row 4 with Row 4- 5 Row 3 and replaces Row 5 with Row 3 + Row 5.

7. (15 points) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Let $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Let $C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Calculate the following:

$$A(BC)$$

$$(AB)C$$

$$C^2$$

$$2A + B - C$$

$$A(B - C)$$

Solution:

$$A(BC) = \begin{pmatrix} 6 & 3 \\ 14 & 7 \end{pmatrix}$$

$$(AB)C = \begin{pmatrix} 6 & 3 \\ 14 & 7 \end{pmatrix}$$

$$C^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$2A + B - C = \begin{pmatrix} 2 & 4 \\ 6 & 9 \end{pmatrix}$$

$$A(B - C) = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}$$

8. (20 points) Prove or disprove that the following sets of vectors form a basis for the indicated vector space:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \text{ in } \mathbb{R}^2$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \text{ in } \mathbb{R}^3$$

Solution:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \text{ in } \mathbb{R}^2$$

These two vectors form a basis for $\mathbb{R}^2$. These two vectors are not scalar multiples of each other, so they are linearly independent. $\mathbb{R}^2$ is two dimensional, so an two linearly independent vectors form a basis.

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix} \text{ in } \mathbb{R}^3$$

These four vectors do not form a basis. A basis consists of linearly independent vectors. $\mathbb{R}^3$ is three dimensional, so no set of more than three vectors can ever be linearly independent in $\mathbb{R}^3$.

9. (25 points) Find all solutions to the equation $A\bar{x} = \bar{b}$ for

$$A = \begin{pmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{pmatrix}$$

$$\bar{b} = \begin{pmatrix}0\\4\end{pmatrix}$$

Solution:

The augmented matrix for this system of equations is:

$$\begin{pmatrix}1 & 2 & 3 & 4 & 0\\5 & 6 & 7 & 8 & 4\end{pmatrix}$$

Put this is reduced row echelon from by performing Gauss-Jordan Elimination:

$$\begin{pmatrix}1 & 0\\-5 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4 & 0\\5 & 6 & 7 & 8 & 4\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4 & 0\\0 & -4 & -8 & -12 & 4\end{pmatrix}$$

$$\begin{pmatrix}1 & \frac{1}{2}\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4 & 0\\0 & -4 & -8 & -12 & 4\end{pmatrix} = \begin{pmatrix}1 & 0 & -1 & -2 & 2\\0 & -4 & -8 & -12 & 4\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\0 & -\frac{1}{4}\end{pmatrix}\begin{pmatrix}1 & 0 & -1 & -2 & 2\\0 & -4 & -8 & -12 & 4\end{pmatrix} = \begin{pmatrix}1 & 0 & -1 & -2 & 2\\0 & 1 & 2 & 3 & -1\end{pmatrix}$$

Solutions to the original equation are the same as solutions to the new equation :

$$\begin{pmatrix}1 & 0 & -1 & -2\\0 & 1 & 2 & 3\end{pmatrix}\bar{x} = \begin{pmatrix}2\\-1\end{pmatrix}$$

Columns 1 and 2 are pivot columns; columns 3 and 4 are free columns, so x_3 and x_4 are the free variables.

A particular solution is obtained by setting $x_3 = x_4 = 0$ and solving for x_1 and x_2.

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

The particular solution is

$$\bar{x}_p = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

The other solutions differ from $\bar{x}_p$ by vectors in the nullspace of A. A basis for the nullspace is given by the "special solutions" obtained by setting one free variable equal to 1, the others equal to 0, and solving for the non-free variables.

$\bar{s}_3$ is the special solution obtained by setting $x_3 = 1$ and $x_4 = 0$ and solving

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $\bar{s}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$

Similarly, s_4 is obtained by setting $x_3 = 0$ and $x_4 = 1$ and solving

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $\bar{s}_4 = \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$

Finally, all solutions are of the form $\bar{x}_p + x_3\bar{s}_3 + x_4\bar{s}_4$, so all solutions to the equation are all vectors:

$$\bar{x} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

where x_3 and x_4 are arbitrary real numbers.

10. (20 points) Prove that a system of 4 linear equations in 5 unknowns has either 0 or infinitely many solutions.

Solution:

There are more unknowns than equations. Either the equations are inconsistent and there are no solutions, or there is a free variable. We may choose this free variable to be any real number and find a corresponding solution, so this is infinitely many solutions.

No new questions beyond this point.