MATH 2270-2 Sample Exam 1 Spring 2012

ANSWERS

1. (5 points) State the *Three Possibilities* for a linear system $A\vec{x} = \vec{b}$.
Answer:
A linear system has either (1) A unique solution, (2) No solution, or (3) Infinitely many solutions.
2. (5 points) Completely describe each operation in the basic Toolkit for solving a linear system (combo, swap, mult).
Answer:
Combo(s,t,c) = Add c times row s to row t
Swap(s,t) = Swap rows s and t
Mult(t,m) = Multiply row t by m (m not zero)
These apply to a matrix. Replace row by equation for systems of equations.
3. (5 points) Can the following system have no solution for some choice of b_1, b_2, b_3?

$$\begin{aligned} x_1 + \quad x_2 + \quad x_3 + \quad x_4 &= b_1 \\ 4x_1 + 2x_2 + 3x_3 \qquad &= b_2 \\ x_3 + \quad x_4 &= b_3 \end{aligned}$$

Answer:
No. There are three lead variables and one free variable. Because the number of lead variables matches the number of equations, then it is impossible to obtain a signal equation.
4. (5 points) True or false? Explain. If A and B are $n \times n$ invertible, then $(AB)^{-1} = A^{-1}B^{-1}$.
Answer:
In general $(AB)^{-1}$ is the product of the inverses in reverse order, $B^{-1}A^{-1}$. There are cases where $B^{-1}A^{-1} \neq A^{-1}B^{-1}$. Start with two invertible matrices C, D such that $CD \neq DC$. Then let $A = C^{-1}$ and $B = D^{-1}$. For example, $C = \begin{pmatrix} 1 & 2 \\ 4 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$.
5. (5 points) True or false? Explain. If square matrices A and B satisfy $AB = I$, then $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ for each vector $\vec{b}$.
Answer:
Both A and B are invertible and inverses of each other. Then $\vec{x} = B\vec{b}$ is the unique solution of $A\vec{x} = \vec{b}$.

6. (5 points) Give an example of a 3×2 matrix A and a frame sequence with three or more frames, starting at A, which proves that the invented system $A\vec{x} = \vec{0}$ has a unique solution.

Answer:

Let A be the augmented matrix of a 2×2 identity matrix and a row of zeros.

7. (5 points) Give an example of a 4×3 matrix A and a frame sequence with three or more frames, starting at A, which proves that the invented system $A\vec{x} = \vec{0}$ has infinitely many solutions.

Answer:

Make the last two rows of A all zeros.

8. (5 points) Let A be a 3×4 matrix. What is the elimination matrix that replaces Row 2 of A with Row 2 minus Row 1 and replaces Row 3 of A by Row 3 minus 2 times Row 1?

Answer:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

9. (5 points) Let A be a 3×3 matrix. Let

$$F = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Assume F is obtained from A by the following sequential row operations: (1) Swap rows 2 and 3; (2) Add -2 times row 2 to row 3; (3) Add 3 times row 1 to row 2; (4) Multiply row 2 by -3. Find A.

Answer:

Apply the inverse elementary operations on F in reverse order: Multiply row 2 by $-1/3$; Add -3 times row 1 to row 2; Add 2 times row 2 to row 3; Swap rows 2 and 3. Then

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -6 & -12 & -\frac{56}{3} \\ -3 & -6 & -\frac{28}{3} \end{pmatrix}.$$

10. (5 points) What is the inverse of the following matrix?

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Answer:

E is the elimination matrix for combo(3,4,-5) (add 5 times Row 3 to Row 4) followed by combo(3,5,1) (add 1 times Row 3 to Row 5). The inverse is the elimination matrix that reverses these steps, combo(3,5,-1) followed by combo(3,4,5).

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

11. (5 points) Describe in words the effect of multiplying

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

on the left of a 4×5 matrix A to get EA.

Answer:

The matrix E is obtained from the identity I by combo(2,3,-3) followed by combo(2,4,2). For EA, the same combo operations are applied to A instead, in the same order.

12. (20 points) Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} 1 & b-c & a \\ 1 & c & -a \\ 2 & b & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -a \\ a \\ a \end{pmatrix}$$

(a). Determine those values of a, b and c such that the system has a unique solution.

(b). Determine those values of a, b and c such that the system has no solution.

(c). Determine those values of a, b and c such that the system has infinitely many solutions.

To save time, don't attempt to solve the equations or to produce a complete frame sequence.

Answer:

No solution for $2c - b = 0$ and $a \neq 0$. Unique solution for $a \neq 0$ and $2c - b \neq 0$. Infinitely many solutions for $a = 0$.

Combo is used to obtain in 5 steps the matrix

$$\begin{pmatrix} 1 & b-c & 0 & -2a \\ 0 & -b+2c & 0 & 4a \\ 0 & 0 & a & a \end{pmatrix}$$

The sequence of steps are documented below for maple.

```
macro(combo=linalg[addrow]);macro(mult=linalg[mulrow]);
macro(swap=linalg[swaprow]);
A:=(a,b,c)->Matrix([[1,b-c,a,-a],[1,c,-a,a],[2,b,a,a]]);
A1:=combo(A(a,b,c),1,2,-1);
A2:=combo(A1,1,3,-2);
A3:=combo(A2,2,3,-1);
A4:=combo(A3,3,2,2);
A5:=combo(A4,3,1,-1);
A5 := Matrix([[1,b-c,0,-2*a],[0,-b+2*c,0,4*a],[0,0,a,a]]);
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13. (15 points) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Let $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Let $C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Calculate the following:

$$A(BC), \quad (AB)C, \quad C^2, \quad 2A+B-C, \quad A(B-C)$$

Answer:

$$A(BC) = \begin{pmatrix} 6 & 3 \\ 14 & 7 \end{pmatrix}$$

$$(AB)C = \begin{pmatrix} 6 & 3 \\ 14 & 7 \end{pmatrix}$$

$$C^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$2A+B-C = \begin{pmatrix} 2 & 4 \\ 6 & 9 \end{pmatrix}$$

$$A(B-C) = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}$$

14. (20 points) Classify the following sets of vectors for (1) Independence or (2) Dependence. For each set of vectors, report whether or not they form a **basis** for the indicated vector space.

$$\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix} \text{ in } \mathbb{R}^2$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix} \text{ in } \mathbb{R}^3$$

Answer:

$$\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix} \text{ in } \mathbb{R}^2$$

These two vectors are not scalar multiples of each other, so they are linearly independent. Because $\mathbb{R}^2$ is two dimensional, any two linearly independent vectors form a basis. These two vectors form a basis for $\mathbb{R}^2$.

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix} \text{ in } \mathbb{R}^3$$

Because $\mathbb{R}^3$ is three dimensional, then no set of more than three vectors can ever be linearly independent in $\mathbb{R}^3$. The set is dependent. These four vectors do not form a basis. A basis consists of three linearly independent vectors. Three such can be determined as the pivot columns of an augmented matrix of the four vectors.

15. (20 points) Find all solutions to the equation $A\bar{x} = \bar{b}$ for

$$A = \begin{pmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{pmatrix}$$

$$\bar{b} = \begin{pmatrix}0\\4\end{pmatrix}$$

Answer:

The augmented matrix for this system of equations is

$$\begin{pmatrix}1 & 2 & 3 & 4 & 0\\5 & 6 & 7 & 8 & 4\end{pmatrix}$$

Find the reduced row echelon form by Gauss-Jordan Elimination, as follows.

$$\begin{pmatrix}1 & 2 & 3 & 4 & 0\\5 & 6 & 7 & 8 & 4\end{pmatrix} \quad \text{Augmented matrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & -4 & -8 & -12 & 4 \end{pmatrix} \quad \text{combo(1,2,-5)}$$

$$\begin{pmatrix} 1 & 0 & -1 & -2 & 2 \\ 0 & 1 & 2 & 3 & -1 \end{pmatrix} \quad \text{mult(2,-1/4)}$$

The last frame, or RREF, implies the system

$$\begin{array}{ccccccc} x_1 & & - & x_3 & - & 2x_4 & = & 2 \\ & x_2 & + & 2x_3 & + & 3x_4 & = & -1 \end{array}$$

The lead variables are x_1, x_2 and the free variables are x_3, x_4. The last frame algorithm introduces invented symbols t_1, t_2. The free variables are set to these symbols, then back-substitute into the lead variable equations of the last frame to obtain the general solution

$$\begin{aligned} x_1 &= 2 + t_1 + 2t_2, \\ x_2 &= -1 - 2t_1 - 3t_2, \\ x_3 &= t_1, \\ x_4 &= t_2. \end{aligned}$$

Strang's *special solutions* $\vec{s}_3$ and $\vec{s}_4$ are the partials of $\vec{x}$ on the invented symbols t_1, t_2. A particular solution $\vec{x}_p$ is obtained by setting all invented symbols to zero. Then

$$\vec{x} = \vec{x}_p + t_1\vec{s}_3 + t_2\vec{s}_4 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

Below is a second solution which follows the methods of Strang's textbook.
Solutions to the original equation are the same as solutions to the last frame,

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix} \vec{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Columns 1 and 2 are pivot columns; columns 3 and 4 are free columns, so x_3 and x_4 are the free variables.
A particular solution is obtained by setting $x_3 = x_4 = 0$ and solving for x_1 and x_2.

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

The particular solution is

$$\bar{x}_p = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

The other solutions differ from $\vec{x}_p$ by vectors in the nullspace of A. A basis for the nullspace is given by the *special solutions* obtained by setting one free variable equal to 1, the others equal to 0, and solving for the non-free variables. This is equivalent to taking partial derivatives on the free variables in the general solution of the system of equations.
Special solution $\vec{s}_3$ is obtained by setting $x_3 = 1$ and $x_4 = 0$ and solving

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $\vec{s}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$
Similarly, $\vec{s_4}$ is obtained by setting $x_3 = 0$ and $x_4 = 1$ and solving

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $\vec{s}_4 = \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$
Finally, all solutions are of the form $\vec{x}_p + x_3\vec{s}_3 + x_4\vec{s}_4$, so all solutions to the equation are all vectors:

$$\vec{x} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

where x_3 and x_4 are arbitrary real numbers.

16. (20 points) Prove that a system of 4 linear equations in 5 unknowns has either no solution or infinitely many solutions.

Answer:

The number of leading variables is at most 4, because there are only 4 equations. Therefore, there is at least one free variable. This means there cannot be a unique solution, and there remains only two possibilities: no solution or infinitely many solutions.

17. (20 points) Let matrix A have 201 rows and 201 columns. All entries of A are zero off the diagonal, except for the number -7 in row 107, column 35. The diagonal entries of A are all one. Describe the inverse of A, in words.

Answer:

It is the same as A except change the -7 to 7. Matrix A is an elementary matrix for combo(35,107,-7). The inverse of A is the elementary matrix for combo(35,107,7).

End of the sample exam questions.