MATH 2270-2 **Final Exam Sample Problems** Spring 2012

ANSWERS

Chapters 1 and 2

1. (5 points) Let A be a 2×2 matrix such that $A\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$. Compute $A\begin{pmatrix}2\\2\end{pmatrix}$.

Answer:

2. (5 points) State (1) the definition of norm, (2) the Cauchy-Schwartz inequality and (3) the triangle inequality, for vectors in $\mathcal{R}^n$.

Answer:

3. (5 points) Define an Elementary Matrix. Display the fundamental matrix multiply equation which summarizes a sequence of swap, combo, multiply operations, transforming a matrix A into a matrix B.

Answer:

4. (5 points) Suppose $A = B(C + D)E$ and all the matrices are invertible. Find an equation for C.

Answer:

5. (5 points) Let $A=\begin{pmatrix}2 & 1\\ 0 & 3\end{pmatrix}$. Show the details of two different methods for finding A^{-1}.

Answer:

6. (5 points) Find a factorization $A = LU$ into lower and upper triangular matrices for the matrix $A=\begin{pmatrix}2 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 2\end{pmatrix}$.

Answer:

7. (5 points) Let Q be a 2×2 matrix with $QQ^T = I$. Prove that Q has columns of unit length and its two columns are orthogonal.

Answer:

Chapters 3, 4

8. (5 points) Let V be a vector space and S a subset of V. State the **Subspace Criterion**, a theorem with three requirements, and conclusion that S is a subspace of V.

Answer:

9. (5 points) Explain how the **span theorem** applies to show that the set S of all linear combinations of the functions $\cosh x, \sinh x$ is a subspace of the vector space V of all continuous functions on $-\infty < x < \infty$.

Answer:

10. (5 points) Write a proof that the subset S of all solutions $\vec{x}$ in $\mathcal{R}^n$ to a homogeneous matrix equation $A\vec{x} = \vec{0}$ is a subspace of $\mathcal{R}^n$. This is called the **kernel theorem**.

Answer:

11. (5 points) Using the subspace criterion, write two hypotheses that imply that a set S in a vector space V is not a subspace of V. The full statement of three such hypotheses is called the **Not a Subspace Theorem**.

Answer:

12. (5 points) Report which columns of A are pivot columns: $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

Answer:

13. (5 points) Find the complete solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

The homogeneous solution $\vec{x}_h$ is a linear combination of Strang's special solutions. Symbol $\vec{x}_p$ denotes a particular solution.

Answer:

14. (5 points) Find the reduced row echelon form of the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

Answer:

15. (5 points) A 10×13 matrix A is given and the homogeneous system $A\vec{x} = \vec{0}$ is transformed to reduced row echelon form. There are 7 lead variables. How many free variables?

Answer:

16. (5 points) The rank of a 10×13 matrix A is 7. Find the nullity of A.

Answer:

17. (5 points) Let S be the subspace of $\mathbb{R}^4$ spanned by the vectors $\bar{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $\bar{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. Find the Gram-Schmidt orthonormal basis of S.

Answer:

18. (5 points) Let the linear transformation T from $\mathcal{R}^3$ to $\mathcal{R}^3$ be defined by its action on three independent vectors: Given a basis

$$T\left(\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}, T\left(\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}, T\left(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}.$$

Find the unique 3×3 matrix A such that T is defined by the matrix multiply equation $T(\vec{x}) = A\vec{x}$.

Answer:

19. (5 points) Determine independence or dependence for the list of vectors

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Answer:

20. (5 points) Use least squares to find the best fit line for the points $(1, 2)$, $(2, 2)$, $(3, 0)$.

Answer:

21. (5 points) Find all solutions to the system of equations

$$\begin{aligned} 2w + 3x + 4y + 5z &= 1 \\ 4w + 3x + 8y + 5z &= 2 \\ 6w + 3x + 8y + 5z &= 1 \end{aligned}$$

Answer:

22. (5 points) The spectral theorem says that a symmetric matrix A can be factored into $A = QDQ^T$ where Q is orthogonal and D is diagonal. Find Q and D for the symmetric matrix $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$.

Answer:

23. (5 points) Show that if B is an invertible matrix and A is similar to B, with $A = PBP^{-1}$, then A is invertible.

Answer:

24. (5 points) Prove that the null space S of an $m \times n$ matrix M is a subspace of $\mathbb{R}^n$. This is called the *Kernel Theorem*, and it is proved from the *Subspace Criterion*.

Answer:

25. (5 points) Let A be an $m \times n$ matrix with independent columns. Prove that $A^T A$ is invertible.

Answer:

26. (5 points) Let A be an $m \times n$ matrix with $A^T A$ invertible. Prove that the columns of A are independent.

Answer:

27. (5 points) Let A be an $m \times n$ matrix. Denote by S_1 the row space of of A and S_2 the column space of A. Prove that $T : S_1 \to S_2$ defined by $T(\vec{x}) = A\vec{x}$ is one-to-one and onto.

Answer:

28. (5 points) Let A be an $m \times n$ matrix and $\vec{v}$ a vector orthogonal to the nullspace of A. Prove that $\vec{v}$ must be in the row space of A.

Answer:

29. (5 points) State the Fundamental Theorem of Linear Algebra. Include **Part 1**: The dimensions of the four subspaces, and **Part 2**: The orthogonality equations for the four subspaces.

Answer:

30. (5 points) Display the equation for the Singular Value Decomposition (SVD), then cite the conditions for each matrix.

Answer:

31. (5 points) Display the equation for the pseudo inverse of A, then define and document each matrix in the product.

Answer: