MATH 2270-003 Final Exam Fall 2010

Part I

1. (5 points) L is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $L\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$. What is $L\begin{pmatrix}2\\2\end{pmatrix}$?

2. (5 points) Consider the plane V in $\mathbb{R}^4$ spanned by the vectors $\bar{v}_1 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}$ and $\bar{v}_2 = \begin{pmatrix}1\\1\\0\\1\end{pmatrix}$.

Give an orthonormal basis for V.

3. (5 points) $\mathcal{A} = \{\begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}4\\4\end{pmatrix}\}$ is a basis for $\mathbb{R}^2$. Let $\bar{v} = \begin{pmatrix}10\\4\end{pmatrix}$. Write $\bar{v}$ in $\mathcal{A}$–coordinates.

4. (5 points) Are the following vectors linearly independent?

$$\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\0\\4\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}$$

5. (5 points) Use least squares approximation to find the best fit line for the points $(1,2)$, $(2,2)$, $(3,0)$.

6. (5 points) Find all solutions to the following system of equations:

$$2w + 3x + 4y + 5z = 1$$
$$4w + 3x + 8y + 5z = 2$$
$$6w + 3x + 8y + 5z = 1$$

7. (5 points) Find an orthogonal matrix Q and diagonal matrix D such that $A = QDQ^T$ for $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$.

8. (5 points) Show that if B is an invertible matrix and A is similar to B then A is invertible.

9. (5 points) M is an $m \times n$ matrix. Prove that the null space of M is a vector subspace of $\mathbb{R}^n$.

10. (5 points) A is an $m \times n$ matrix of rank r. Show that the rank of $A^T A$ is also r. Conclude that $A^T A$ is invertible if and only if the columns of A are linearly independent.

Part II

State the Fundamental Theorem of Linear Algebra, Parts 1 and 2, and the Singular Value Decomposition (SVD) Theorem. Explain, in plain English, what they mean and how they complement each other. Give an explicit (actual numbers in the entries) example of a matrix M such that each of the four fundamental subspaces is nontrivial, and such that the two bases in the SVD are not the same. Compute the SVD of M and identify orthonormal bases for the four fundamental subspaces.