

Vector Spaces and Subspaces

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Vector Space V

It is a **data set** V plus a **toolkit** of eight (8) algebraic properties. The data set consists of packages of data items, called **vectors**, denoted \vec{X} , \vec{Y} below.

Closure	The operations $\vec{X} + \vec{Y}$ and $k\vec{X}$ are defined and result in a new vector which is also in the set V .	
Addition	$\vec{X} + \vec{Y} = \vec{Y} + \vec{X}$	commutative
	$\vec{X} + (\vec{Y} + \vec{Z}) = (\vec{Y} + \vec{X}) + \vec{Z}$	associative
	Vector $\vec{0}$ is defined and $\vec{0} + \vec{X} = \vec{X}$	zero
	Vector $-\vec{X}$ is defined and $\vec{X} + (-\vec{X}) = \vec{0}$	negative
Scalar multiply	$k(\vec{X} + \vec{Y}) = k\vec{X} + k\vec{Y}$	distributive I
	$(k_1 + k_2)\vec{X} = k_1\vec{X} + k_2\vec{X}$	distributive II
	$k_1(k_2\vec{X}) = (k_1k_2)\vec{X}$	distributive III
	$1\vec{X} = \vec{X}$	identity

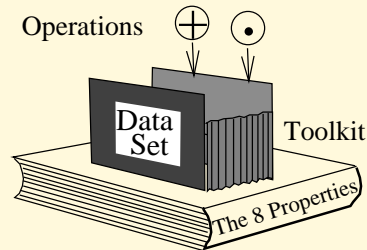


Figure 1. A *Vector Space* is a data set, operations \oplus and \odot , and the 8-property toolkit.

Definition of Subspace

A **subspace** S of a vector space V is a nonvoid subset of V which under the operations $+$ and \cdot of V forms a vector space in its own right.

Subspace Criterion

Let S be a subset of V such that

1. Vector $\mathbf{0}$ is in S .
2. If \vec{X} and \vec{Y} are in S , then $\vec{X} + \vec{Y}$ is in S .
3. If \vec{X} is in S , then $c\vec{X}$ is in S .

Then S is a subspace of V .

Items **2**, **3** can be summarized as *all linear combinations of vectors in S are again in S* . In proofs using the criterion, items 2 and 3 may be replaced by

$$c_1\vec{X} + c_2\vec{Y} \text{ is in } S.$$

Subspaces are Working Sets

We call a subspace S of a vector space V a **working set**, because the purpose of identifying a subspace is to shrink the original data set V into a smaller data set S , customized for the application under study.

A Key Example. Let V be ordinary space R^3 and let S be the plane of action of a planar kinematics experiment. The data set for the experiment is all 3-vectors v in V collected by a data recorder. Detected and recorded is the 3D position of a particle which has been constrained to a plane. The plane of action S is computed as a homogeneous equation like $2x+3y+1000z=0$, the equation of a plane, from the recorded data set in V . After least squares is applied to find the optimal equation for S , then S replaces the larger data set V . The customized smaller set S is the *working set* for the kinematics problem.

The Kernel Theorem

Theorem 1 (Kernel Theorem)

Let V be one of the vector spaces \mathbb{R}^n and let A be an $m \times n$ matrix. Define a smaller set S of data items in V by the kernel equation

$$S = \{\mathbf{x} : \mathbf{x} \text{ in } V, \quad A\mathbf{x} = \mathbf{0}\}.$$

Then S is a subspace of V .

In particular, operations of addition and scalar multiplication applied to data items in S give answers back in S , and the 8-property toolkit applies to data items in S .

Proof: Zero is in V because $A\mathbf{0} = \mathbf{0}$ for any matrix A . To verify the subspace criterion, we verify that $\mathbf{z} = c_1\mathbf{x} + c_2\mathbf{y}$ for \mathbf{x} and \mathbf{y} in V also belongs to V . The details:

$$\begin{aligned} A\mathbf{z} &= A(c_1\mathbf{x} + c_2\mathbf{y}) \\ &= A(c_1\mathbf{x}) + A(c_2\mathbf{y}) \\ &= c_1A\mathbf{x} + c_2A\mathbf{y} \\ &= c_1\mathbf{0} + c_2\mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

Because $A\mathbf{x} = A\mathbf{y} = \mathbf{0}$, due to \mathbf{x}, \mathbf{y} in V .

Therefore, $A\mathbf{z} = \mathbf{0}$, and \mathbf{z} is in V .

The proof is complete.

Not a Subspace Theorem

Theorem 2 (Testing S not a Subspace)

Let V be an abstract vector space and assume S is a subset of V . Then S is not a subspace of V provided one of the following holds.

- (1) The vector 0 is not in S .
- (2) Some x and $-x$ are not both in S .
- (3) Vector $x + y$ is not in S for some x and y in S .

Proof: The theorem is justified from the *Subspace Criterion*.

1. The criterion requires 0 is in S .
2. The criterion demands cx is in S for all scalars c and all vectors x in S .
3. According to the subspace criterion, the sum of two vectors in S must be in S .

Definition of Independence and Dependence

A list of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in a vector space V are said to be **independent** provided every linear combination of these vectors is uniquely represented. **Dependent** means **not independent**.

Unique representation

An equation

$$a_1\mathbf{v}_1 + \dots + a_k\mathbf{v}_k = b_1\mathbf{v}_1 + \dots + b_k\mathbf{v}_k$$

implies matching coefficients: $a_1 = b_1, \dots, a_k = b_k$.

Independence Test

Form the system in unknowns c_1, \dots, c_k

$$c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k = \mathbf{0}.$$

Solve for the unknowns [how to do this depends on V]. Then the vectors are independent if and only if the unique solution is $c_1 = c_2 = \dots = c_k = 0$.

Independence test for two vectors v_1, v_2

In an abstract vector space V , form the equation

$$c_1v_1 + c_2v_2 = 0.$$

Solve this equation for c_1, c_2 .

Then v_1, v_2 are independent in V if and only if the system has unique solution $c_1 = c_2 = 0$.

Geometry and Independence

- One fixed vector is independent if and only if it is nonzero.
- Two fixed vectors are independent if and only if they form the edges of a parallelogram of positive area.
- Three fixed vectors are independent if and only if they are the edges of a parallelepiped of positive volume.

In an abstract vector space V , two vectors [two data packages] are independent if and only if one is not a scalar multiple of the other.

Illustration

Vectors $\mathbf{v}_1 = \cos x$ and $\mathbf{v}_2 = \sin x$ are two data packages [graphs] in the vector space V of continuous functions. They are independent because one graph is not a scalar multiple of the other graph.

An Illustration of the Independence Test

Two column vectors are tested for independence by forming the system of equations $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$, e.g,

$$c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is a homogeneous system $\mathbf{A}\mathbf{c} = \mathbf{0}$ with

$$\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The system $\mathbf{A}\mathbf{c} = \mathbf{0}$ can be solved for \mathbf{c} by frame sequence methods. Because $\text{rref}(\mathbf{A}) = \mathbf{I}$, then $c_1 = c_2 = 0$, which verifies independence.

If the system $\mathbf{A}\mathbf{c} = \mathbf{0}$ is square, then $\det(\mathbf{A}) \neq 0$ applies to test independence.

There is **no chance to use determinants** when the system is not square, e.g., consider the homogeneous system

$$c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It has vector-matrix form $\mathbf{A}\mathbf{c} = \mathbf{0}$ with 3×2 matrix \mathbf{A} , for which $\det(\mathbf{A})$ is undefined.

Rank Test

In the vector space \mathbf{R}^n , the independence test leads to a system of n linear homogeneous equations in k variables c_1, \dots, c_k . The test requires solving a matrix equation $\mathbf{A}\mathbf{c} = \mathbf{0}$. The signal for independence is **zero free variables**, or nullity zero, or equivalently, maximal rank. To justify the various statements, we use the relation $\text{nullity}(\mathbf{A}) + \text{rank}(\mathbf{A}) = k$, where k is the column dimension of \mathbf{A} .

Theorem 3 (Rank-Nullity Test)

Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be k column vectors in \mathbf{R}^n and let \mathbf{A} be the augmented matrix of these vectors. The vectors are independent if $\text{rank}(\mathbf{A}) = k$ and dependent if $\text{rank}(\mathbf{A}) < k$. The conditions are equivalent to $\text{nullity}(\mathbf{A}) = 0$ and $\text{nullity}(\mathbf{A}) > 0$, respectively.

Determinant Test

In the unusual case when the system arising in the independence test can be expressed as $\mathbf{A}\mathbf{c} = \mathbf{0}$ and \mathbf{A} is square, then $\det(\mathbf{A}) = 0$ detects dependence, and $\det(\mathbf{A}) \neq 0$ detects independence. The reasoning is based upon the adjugate formula $\mathbf{A}^{-1} = \mathbf{adj}(\mathbf{A}) / \det(\mathbf{A})$, valid exactly when $\det(\mathbf{A}) \neq 0$.

Theorem 4 (Determinant Test)

Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be n column vectors in \mathbf{R}^n and let \mathbf{A} be the augmented matrix of these vectors. The vectors are independent if $\det(\mathbf{A}) \neq 0$ and dependent if $\det(\mathbf{A}) = 0$.

Sampling Test

Let functions f_1, \dots, f_n be given and let x_1, \dots, x_n be distinct x -sample values. Define

$$A = \begin{pmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \vdots & \vdots & \cdots & \vdots \\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n) \end{pmatrix}.$$

Then $\det(A) \neq 0$ implies f_1, \dots, f_n are independent functions.

Proof

We'll do the proof for $n = 2$. Details are similar for general n . Assume $c_1 f_1 + c_2 f_2 = 0$. Then for all x , $c_1 f_1(x) + c_2 f_2(x) = 0$. Choose $x = x_1$ and $x = x_2$ in this relation to get $A\mathbf{c} = \mathbf{0}$, where \mathbf{c} has components c_1, c_2 . If $\det(A) \neq 0$, then A^{-1} exists, and this in turn implies $\mathbf{c} = A^{-1}A\mathbf{c} = \mathbf{0}$. We conclude f_1, f_2 are independent.

Wronskian Test

Let functions f_1, \dots, f_n be given and let x_0 be a given point. Define

$$W = \begin{pmatrix} f_1(x_0) & f_2(x_0) & \cdots & f_n(x_0) \\ f_1'(x_0) & f_2'(x_0) & \cdots & f_n'(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(x_0) & f_2^{(n-1)}(x_0) & \cdots & f_n^{(n-1)}(x_0) \end{pmatrix}.$$

Then $\det(W) \neq 0$ implies f_1, \dots, f_n are independent functions. The matrix W is called the **Wronskian Matrix** of f_1, \dots, f_n and $\det(W)$ is called the **Wronskian determinant**.

Proof

We'll do the proof for $n = 2$. Details are similar for general n . Assume $c_1 f_1 + c_2 f_2 = 0$. Then for all x , $c_1 f_1(x) + c_2 f_2(x) = 0$ and $c_1 f_1'(x) + c_2 f_2'(x) = 0$. Choose $x = x_0$ in this relation to get $Wc = 0$, where c has components c_1, c_2 . If $\det(W) \neq 0$, then W^{-1} exists, and this in turn implies $c = W^{-1}Wc = 0$. We conclude f_1, f_2 are independent.

Atoms

Definition. A **base atom** is one of the functions

$$1, \quad e^{ax}, \quad \cos bx, \quad \sin bx, \quad e^{ax} \cos bx, \quad e^{ax} \sin bx$$

where $b > 0$. Define an **atom** for integers $n \geq 0$ by the formula

$$\mathbf{atom} = x^n(\mathbf{base\ atom}).$$

The powers $1, x, x^2, \dots$ are atoms (multiply base atom 1 by x^n). Multiples of these powers by $\cos bx, \sin bx$ are also atoms. Finally, multiplying all these atoms by e^{ax} expands and completes the list of atoms.

Alternatively, an **atom** is a function with coefficient 1 obtained as the real or imaginary part of the complex expression

$$x^n e^{ax} (\cos bx + i \sin bx).$$

Illustration

We show the powers $1, x, x^2, x^3$ are independent atoms by applying the Wronskian Test:

$$W = \begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 0 & 1 & 2x_0 & 3x_0^2 \\ 0 & 0 & 2 & 6x_0 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

Then $\det(W) = 12 \neq 0$ implies the functions $1, x, x^2, x^3$ are linearly independent.

Subsets of Independent Sets are Independent

Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ make an independent set and consider the subset $\mathbf{v}_1, \mathbf{v}_2$. If

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$$

then also

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

where $c_3 = 0$. Independence of the larger set implies $c_1 = c_2 = c_3 = 0$, in particular, $c_1 = c_2 = 0$, and then $\mathbf{v}_1, \mathbf{v}_2$ are independent.

Theorem 5 (Subsets and Independence)

- A non-void subset of an independent set is also independent.
- Non-void subsets of dependent sets can be independent or dependent.

Atoms and Independence

Theorem 6 (Independence of Atoms)

Any list of distinct atoms is linearly independent.

Unique Representation

The theorem is used to extract equations from relations involving atoms. For instance:

$$(c_1 - c_2) \cos x + (c_1 + c_3) \sin x + c_1 + c_2 = 2 \cos x + 5$$

implies

$$c_1 - c_2 = 2,$$

$$c_1 + c_3 = 0,$$

$$c_1 + c_2 = 5.$$

Atoms and Differential Equations

It is known that solutions of linear constant coefficient differential equations of order n and also systems of linear differential equations with constant coefficients have a general solution which is a linear combination of atoms.

- The harmonic oscillator $y'' + b^2y = 0$ has general solution $y(x) = c_1 \cos bx + c_2 \sin bx$. This is a linear combination of the two atoms $\cos bx$, $\sin bx$.
- The third order equation $y''' + y' = 0$ has general solution $y(x) = c_1 \cos x + c_2 \sin x + c_3$. The solution is a linear combination of the independent atoms $\cos x$, $\sin x$, 1 .
- The linear dynamical system $x'(t) = y(t)$, $y'(t) = -x(t)$ has general solution $x(t) = c_1 \cos t + c_2 \sin t$, $y(t) = -c_1 \sin t + c_2 \cos t$, each of which is a linear combination of the independent atoms $\cos t$, $\sin t$.