Applications of Systems of Differential Equations

- Brine Tank Cascade
- Cascade Model
- Recycled Brine Tank Cascade
- Recycled Cascade Model
Brine Tank Cascade

Let brine tanks $A$, $B$, $C$ be given of volumes $20$, $40$, $60$, respectively, as in Figure 1.

Assumptions and Notation

- It is supposed that fluid enters tank $A$ at rate $r$, drains from $A$ to $B$ at rate $r$, drains from $B$ to $C$ at rate $r$, then drains from tank $C$ at rate $r$. Hence the volumes of the tanks remain constant. Let $r = 10$, to illustrate the ideas.

- Uniform stirring of each tank is assumed, which implies uniform salt concentration throughout each tank.

- Let $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the amount of salt at time $t$ in each tank. We suppose added to tank $A$ water containing no salt. Therefore, the salt in all the tanks is eventually lost from the drains.
The cascade is modeled by the chemical balance law

\[
\text{rate of change} = \text{input rate} - \text{output rate}.
\]

Application of the balance law results in the triangular differential system

\[
\begin{align*}
x'_1 &= -\frac{1}{2}x_1, \\
x'_2 &= \frac{1}{2}x_1 - \frac{1}{4}x_2, \\
x'_3 &= \frac{1}{4}x_2 - \frac{1}{6}x_3.
\end{align*}
\]
Cascade Model Solution

The solution is justified by the integrating factor method for first order scalar differential equations.

\[
x_1(t) = x_1(0)e^{-t/2},
\]
\[
x_2(t) = -2x_1(0)e^{-t/2} + (x_2(0) + 2x_1(0))e^{-t/4},
\]
\[
x_3(t) = \frac{3}{2}x_1(0)e^{-t/2} - 3(x_2(0) + 2x_1(0))e^{-t/4}

+ (x_3(0) - \frac{3}{2}x_1(0) + 3(x_2(0) + 2x_1(0)))e^{-t/6}.
\]
Recycled Brine Tank Cascade

Let brine tanks $A$, $B$, $C$ be given of volumes 60, 30, 60, respectively, as in Figure 2.

![Figure 2. Three brine tanks in cascade with recycling.](image)

Assumptions and Notation

- Suppose that fluid drains from tank $A$ to $B$ at rate $r$, drains from tank $B$ to $C$ at rate $r$, then drains from tank $C$ to $A$ at rate $r$. The tank volumes remain constant due to constant recycling of fluid. For purposes of illustration, let $r = 10$.

- Uniform stirring of each tank is assumed, which implies **uniform salt concentration** throughout each tank.

- Let $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the amount of salt at time $t$ in each tank. No salt is lost from the system, due to recycling.
Recycled Cascade Model

Using compartment analysis, the recycled cascade is modeled by the non-triangular system

\[
\begin{align*}
x'_1 &= -\frac{1}{6} x_1 + \frac{1}{6} x_3, \\
x'_2 &= \frac{1}{6} x_1 - \frac{1}{3} x_2, \\
x'_3 &= \frac{1}{3} x_2 - \frac{1}{6} x_3.
\end{align*}
\]
Recycled Cascade Solution

\[ x_1(t) = c_1 + (c_2 - 2c_3)e^{-t/3} \cos(t/6) + (2c_2 + c_3)e^{-t/3} \sin(t/6), \]
\[ x_2(t) = \frac{1}{2}c_1 + (-2c_2 - c_3)e^{-t/3} \cos(t/6) + (c_2 - 2c_3)e^{-t/3} \sin(t/6), \]
\[ x_3(t) = c_1 + (c_2 + 3c_3)e^{-t/3} \cos(t/6) + (-3c_2 + c_3)e^{-t/3} \sin(t/6). \]

- At infinity, \( x_1 = x_3 = c_1, x_2 = c_1/2 \). The meaning is that the total amount of salt is uniformly distributed in the tanks, in the ratio \( 2 : 1 : 2 \).

- The solution of the system was found by the eigenanalysis method. It can also be found by the resolvent method in Laplace theory.