

## Examples: Solving $n$ th Order Equations

- Atoms and Base Atoms
- L. Euler's Theorem
- The Atom List
- First Order. Solve  $2y' + 5y = 0$ .
- Second Order. Solve the equations  $y'' + 2y' + y = 0$ ,  $y'' + 3y' + 2y = 0$  and  $y'' + 2y' + 5y = 0$ .
- Third Order. Solve  $y''' - y' = 0$  and  $y''' - y'' = 0$ .
- Fourth Order. Solve  $y^{(4)} - y^{(2)} = 0$ .

## Atoms

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Assume symbols  $a$  and  $b$  are real constants, with  $a \neq 0$  and  $b > 0$ .

A **base atom** is one of the functions

$$1, e^{ax}, \cos bx, \sin bx, e^{ax} \cos bx, e^{ax} \sin bx.$$

An **atom** is a base atom multiplied by a power  $x^m$ , where  $m \geq 0$  is an integer.

## Examples

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Atoms:  $1, \cos(3x/2), x^3, x \cos(2x), x^2 e^{2x} \sin(3x), \cos(\pi x), e^{e^2 x}$

Not an atom:  $2, -x, x^{3/2}, \tan x, \sin(x^2), e^{x^2}, \frac{x}{1+x^2}, \ln|x|, \sinh x$

## Euler's Theorem

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### Theorem 1 (L. Euler)

The function  $y = x^m e^{r_1 x}$  is a solution of a constant-coefficient linear homogeneous differential equation of the  $n$ th order if and only if  $(r - r_1)^{m+1}$  divides the characteristic polynomial.

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Euler's theorem is used to construct solutions of the  $n$ th order differential equation. The solutions so constructed are  $n$  distinct atoms, hence independent. Picard's theorem implies the list of atoms is a basis for the solution space.

## Euler's Theorem in Words

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To construct solutions of homogeneous constant-coefficient differential equations, use Euler's Theorem as follows.

- Find the roots of the characteristic equation.
  - For each real root  $r$ , the exponential solution  $e^{rx}$  is a base atom solution.
  - For each complex conjugate pair of roots  $a \pm bi$ ,  $b > 0$ , the functions  $e^{ax} \cos bx$ ,  $e^{ax} \sin bx$  are base atom solutions.
- Multiply each base atom by the powers  $1, x, x^2, \dots, x^{k-1}$ , where  $k$  is the multiplicity of the root. The atoms so found are a basis of solutions for the differential equation.

## The Atom List

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1. If  $r_1$  is a real root, then the base atom for  $r_1$  is  $e^{r_1 x}$ . The root  $r_1$  has multiplicity  $k$  provided factor  $(r - r_1)^k$  divides the characteristic polynomial, but factor  $(r - r_1)^{k+1}$  does not. Multiply the base atom by powers  $1, x, \dots, x^{k-1}$  (a total of  $k$  terms) to obtain the atom list

$$e^{r_1 x}, xe^{r_1 x}, \dots, x^{k-1} e^{r_1 x}.$$

2. If  $r_1 = a + ib$  ( $b > 0$  assumed) and its conjugate  $r_2 = a - ib$  are roots of the characteristic equation, then the base atoms for this pair of roots (both  $r_1$  and  $r_2$  counted) are

$$e^{ax} \cos bx, \quad e^{ax} \sin bx.$$

For a root of multiplicity  $k$ , these real atoms are multiplied by atoms  $1, \dots, x^{k-1}$  to obtain the complete list of  $2k$  atoms

$$\begin{aligned} &e^{ax} \cos bx, \quad xe^{ax} \cos bx, \quad \dots, \quad x^{k-1} e^{ax} \cos bx, \\ &e^{ax} \sin bx, \quad xe^{ax} \sin bx, \quad \dots, \quad x^{k-1} e^{ax} \sin bx. \end{aligned}$$

**1 Example (First Order)** Solve  $2y' + 5y = 0$  by Euler's method, verifying  $y_h = c_1 e^{-5x/2}$ .

### Solution

$2y' + 5y = 0$       Given differential equation.

$2r + 5 = 0$       Characteristic equation. Use the shortcut  $y^{(n)} \rightarrow r^n$ .

$r = -5/2$       Exactly one real root.

Atom =  $e^{-5x/2}$       For a real root  $r$ , the base atom is  $e^{rx}$ .

$y_h = c_1 e^{-5x/2}$       The general solution  $y_h$  is written by multiplying the atom list by constant  $c_1$ .

**2 Example (Second Order I)** Solve  $y'' + 2y' + y = 0$  by Euler's method, showing that  $y_h = c_1e^{-x} + c_2xe^{-x}$ .

**Solution**

$$y'' + 2y' + y = 0$$

Given differential equation.

$$r^2 + 2r + 1 = 0$$

Characteristic equation. Use shortcut  $y^{(n)} \rightarrow r^n$ .

$$r = -1, -1$$

Exactly two real roots.

$$\text{Base Atom} = e^{-x}$$

For a real root  $r_1$ , the base atom is  $e^{r_1x}$ .

$$\text{Atoms} = e^{-x}, xe^{-x}$$

For a root of multiplicity 2, multiply the base atom by powers 1,  $x$ , to create 2 atoms.

$$y_h = c_1e^{-x} + c_2xe^{-x}$$

The general solution  $y_h$  is written by multiplying the atom list by constants  $c_1, c_2$ .

**3 Example (Second Order II)** Solve  $y'' + 3y' + 2y = 0$  by Euler's method, showing  $y_h = c_1e^{-x} + c_2e^{-2x}$ .

**Solution**

$$y'' + 3y' + 2y = 0$$

$$r^2 + 3r + 2 = 0$$

$$r = -1, -2$$

$$\text{Base Atoms} = \begin{matrix} e^{-x} \\ e^{-2x} \end{matrix},$$

$$\text{Atoms} = e^{-x}, e^{-2x}$$

$$y_h = c_1e^{-x} + c_2e^{-2x}$$

Given differential equation.

Characteristic equation. Use shortcut  $y^{(n)} \rightarrow r^n$ .

Factorization  $(r + 2)(r + 1) = 0$ .

For a real root  $r_1$ , the base atom is  $e^{r_1x}$ .

For a root of multiplicity 1, multiply the base atom by 1, to create one term. The number of terms created from one base atom always equals the multiplicity of the root.

The general solution  $y_h$  is written by multiplying the atom list by constants  $c_1, c_2$ .

**4 Example (Second Order III)** Solve  $y'' + 2y' + 5y = 0$  by Euler's method, showing  $y_h = c_1 e^{-x} \cos 2x + c_2 x e^{-x} \sin 2x$ .

**Solution**

$$y'' + 2y' + 5y = 0$$

$$r^2 + 2r + 5 = 0$$

$$r = -1 + 2i, -1 - 2i$$

$$\text{Base Atoms} = \begin{matrix} e^{-x} \cos 2x, \\ e^{-x} \sin 2x \end{matrix}$$

$$\text{Atoms} = e^{-x} \cos 2x, e^{-x} \sin 2x$$

$$y_h = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x$$

Given differential equation.

Characteristic equation.

Shortcut  $y^{(n)} \rightarrow r^n$  used.

Factorization  $(r + 1)^2 + 4 = 0$  implies  $r + 1 = \pm 2\sqrt{-1} = \pm 2i$ .

For complex conjugate roots  $r = a \pm ib$ , the base atoms are  $e^{ax} \cos bx$  and  $e^{ax} \sin bx$ .

Multiply the base atoms by 1, because the multiplicity of root  $1 + 2i$  is 1. In general, multiply by 1,  $x$ ,  $\dots$ ,  $x^{k-1}$ , where  $k$  is the root multiplicity.

Multiply the atom list by constants  $c_1$ ,  $c_2$ . This is the general solution.

**5 Example (Third Order I)** Solve  $y''' - y' = 0$  by Euler's method, showing  $y_h = c_1 + c_2e^x + c_3e^{-x}$ .

**Solution**

$$y''' - y' = 0$$

$$r^3 - r = 0$$

$$r = 0, 1, -1$$

$$\text{Base Atoms} = 1, e^{-x}, e^x$$

$$\text{Atoms} = 1, e^{-x}, e^x$$

$$y_h = c_1 + c_2e^{-x} + c_3e^x$$

Given differential equation.

Characteristic equation. Use  $y^{(n)} \rightarrow r^n$ .

Factorization  $r(r + 1)(r - 1) = 0$ .

For a real root  $r_1$ , the base atom is  $e^{r_1x}$ .

Each root has multiplicity 1. Multiply each base atom by 1. Generally, multiply by  $1, x, \dots, x^{k-1}$ , where  $k$  is the root multiplicity.

The general solution  $y_h$  is written by multiplying the atom list by constants  $c_1, c_2, c_3$ .

**6 Example (Third Order II)** Solve  $y''' - y'' = 0$  by Euler's method, showing  $y_h = c_1 + c_2x + c_3e^x$ .

**Solution**

$$y''' - y'' = 0$$

$$r^3 - r^2 = 0$$

$$r = 0, 0, 1$$

$$\text{Base Atoms} = 1, e^x$$

$$\text{Atoms} = 1, x, e^x$$

$$y_h = c_1 + c_2x + c_3e^x$$

Given differential equation.

Characteristic equation. Use  $y^{(n)} \rightarrow r^n$ .

Factorization  $r^2(r - 1) = 0$ .

For a real root  $r_1$ , the base atom is  $e^{r_1x}$ . Then the base atoms are  $e^{0x}$ ,  $e^{1x}$ , written as  $1$ ,  $e^x$ .

Because  $r = 0$  has multiplicity 2, then multiply base atom  $1 (=e^{0x})$  by powers  $1$ ,  $x$ . Root  $r = 1$  with multiplicity 1 implies base atom  $e^x$  is multiplied by 1. The total number of atoms created is  $2 + 1 = 3$ , which is the sum of the root multiplicities.

The general solution  $y_h$  is written by multiplying the atom list by constants  $c_1$ ,  $c_2$ ,  $c_3$ .

**7 Example (Fourth Order)** Solve  $y^{(4)} - y^{(2)} = 0$  by Euler's method, showing that the general solution is  $y_h = c_1 + c_2x + c_3e^x + c_4e^{-x}$ .

### Solution

$$y^{(4)} - y^{(2)} = 0$$

$$r^4 - r^2 = 0$$

$$r = 0, 0, 1, -1$$

$$\text{Base Atoms} = 1, e^x, e^{-x}$$

$$\text{Atoms} = 1, x, e^x, e^{-x}$$

$$y_h = c_1 + c_2x + c_3e^x + c_4e^{-x}$$

Given differential equation.

Characteristic equation. Use  $y^{(n)} \rightarrow r^n$ .

Factorization  $r^2(r - 1)(r + 1) = 0$ .

For a real root  $r_1$ , the base atom is  $e^{r_1x}$ . The base atoms  $e^{0x}$ ,  $e^{1x}$ ,  $e^{-1x}$  are written  $1$ ,  $e^x$ ,  $e^{-x}$ .

Multiply the multiplicity 2 base atom  $1$  by powers  $1$ ,  $x$  and each of the multiplicity 1 base atoms  $e^x$ ,  $e^{-x}$  by  $1$ . The total number of atoms created is  $2 + 1 + 1 = 4$ , which is the sum of the root multiplicities.

Multiply the atom list by constants  $c_1$ ,  $c_2$ ,  $c_3$ ,  $c_4$  and add. This is the general solution.