Transform Properties

Collected here are the major theorems for the manipulation of Laplace transform tables.

- Lerch’s Cancelation Law
- Linearity
- The Parts Rule (t-Derivative Rule)
- The $t$-Integral Rule
- The $s$-Differentiation Rule
- First Shifting Rule
- Second Shifting Rule
- Periodic Function Rule
- Convolution Rule
Theorem 1 (Lerch)
If $f_1(t)$ and $f_2(t)$ are continuous, of exponential order and

$$
\int_0^\infty f_1(t)e^{-st}dt = \int_0^\infty f_2(t)e^{-st}dt
$$

for all $s > s_0$, then for $t \geq 0$,

$$
f_1(t) = f_2(t).
$$

The result is remembered as the cancelation law

$$
L(f_1(t)) = L(f_2(t)) \text{ implies } f_1(t) = f_2(t).
$$
Theorem 2 (Linearity)
The Laplace transform has these inherited integral properties:

(a) $L(f(t) + g(t)) = L(f(t)) + L(g(t))$,
(b) $L(cf(t)) = cL(f(t))$. 
Theorem 3 (The Parts Rule)
Let \( y(t) \) be continuous, of exponential order and let \( y'(t) \) be piecewise continuous on \( t \geq 0 \). Then \( L(y'(t)) \) exists and

\[
L(y'(t)) = sL(y(t)) - y(0).
\]

Theorem 4 (The \( t \)-Integral Rule)
Let \( g(t) \) be of exponential order and continuous for \( t \geq 0 \). Then

\[
L\left( \int_0^t g(x) \, dx \right) = \frac{1}{s}L(g(t)).
\]

- The parts rule is also called the \( t \)-derivative rule. It is used to remove derivatives \( y' \) from Laplace equations.
- The two rules are related by \( y(t) = \int_0^t g(x) \, dx \).
Theorem 5 (The $s$-Differentiation Rule)
Let $f(t)$ be of exponential order. Then

$$L(tf(t)) = -\frac{d}{ds}L(f(t)).$$

The rule says that each factor of $t$ in the integrand of a Laplace integral can be crossed out provided an operation $-d/ds$ is inserted in front of the integral. It is remembered as

*multiplying by $(-t)$ differentiates the transform.*
Theorem 6 (First Shifting Rule)
Let $f(t)$ be of exponential order and $-\infty < a < \infty$. Then

$$L(e^{at}f(t)) = L(f(t)) \big|_{s \mapsto (s-a)}.$$

The rule says that an exponential factor $e^{at}$ in the integrand can be crossed out, provided this action is compensated by replacing $s$ by $s - a$ in the answer. It is remembered as

*multiplying by $e^{at}$ shifts the transform $s \rightarrow s - a$.*
Heaviside Step

The Step function is defined by \( \text{step}(t) = 1 \) for \( t \geq 0 \) and \( \text{step}(t) = 0 \) for \( t < 0 \). It is the same as the unit step \( u(t) \) and the Heaviside function \( H(t) \). Then \( \text{step}(t - a) \) is the step function shifted from the origin to location \( t = a \),

\[
\text{step}(t - a) = \begin{cases} 
1 & a \leq t < \infty, \\
\text{otherwise}.
\end{cases}
\]

The function \( \text{pulse} \) is a finite interval step function defined by

\[
\text{pulse}(t, a, b) = \begin{cases} 
1 & a \leq t < b, \\
0 & \text{otherwise}
\end{cases} = \text{step}(t - a) - \text{step}(t - b).
\]

Maple Worksheet Definitions

\[
\begin{align*}
\text{step} & := \text{unapply}(\text{piecewise}(t \geq 0, 1, 0), t); \\
\text{pulse} & := \text{unapply}(\text{step}(t-a)-\text{step}(t-b), (t,a,b));
\end{align*}
\]
Step Function Shifting Rule

**Theorem 7 (Second Shifting Rule)**
Let \( f(t) \) and \( g(t) \) be of exponential order and assume \( a \geq 0 \). Let \( u(t) = \text{step}(t) \). Then

(a) \( L(f(t - a)u(t - a)) = e^{-as}L(f(t)) \),
(b) \( L(g(t)u(t - a)) = e^{-as}L(g(t + a)) \).

The relations are used to manipulate Laplace equations that arise in differential equations with piecewise defined inputs. Electrical engineering has many such examples.
Theorem 8 (Periodic Function Rule)
Let $f(t)$ be of exponential order and satisfy $f(t + P) = f(t)$. Then

$$L(f(t)) = \frac{\int_0^P f(t)e^{-st}dt}{1 - e^{-Ps}}.$$
Some Engineering Functions

Tabulated here are common periodic functions used in engineering applications.

\[ L(\text{floor}(t/a)) = \frac{e^{-as}}{s(1 - e^{-as})} \]

Staircase function,
\( \text{floor}(x) = \) greatest integer \( \leq x \).

\[ L(\text{sqw}(t/a)) = \frac{1}{s} \tanh\left(\frac{as}{2}\right) \]

Square wave,
\( \text{sqw}(x) = (-1)^\text{floor}(x) \).

\[ L(\text{trw}(t/a)) = \frac{1}{s^2} \tanh\left(\frac{as}{2}\right) \]

Triangular wave,
\( \text{trw}(x) = \int_0^x \text{sqw}(r)dr \).
Theorem 9 (Convolution Rule)
Let \( f(t) \) and \( g(t) \) be of exponential order. Then

\[
L(f(t))L(g(t)) = L \left( \int_0^t f(x)g(t-x)\,dx \right).
\]

An example:

\[
\frac{1}{s^2} \cdot \frac{1}{s-2} = L(t)L(e^{2t})
\]

\[
= L \left( \int_0^t xe^{2(t-x)}\,dx \right)
\]

\[
= L \left( \frac{1}{3}e^{2t} - \frac{1}{2}t - \frac{1}{4} \right)
\]