Basic Laplace Theory

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The integral
\[ \int_0^\infty g(t)e^{-st} dt \]
is called the Laplace integral of the function \( g(t) \). It is defined by
\[ \int_0^\infty g(t)e^{-st} dt \equiv \lim_{N \to \infty} \int_0^N g(t)e^{-st} dt \]
and it depends on variable \( s \). The ideas will be illustrated for \( g(t) = 1 \), \( g(t) = t \) and \( g(t) = t^2 \). Results appear in Table 1 infra.
Laplace Integral or Direct Laplace Transform

The Laplace integral or the direct Laplace transform of a function \( f(t) \) defined for \( 0 \leq t < \infty \) is the ordinary calculus integration problem

\[
\int_0^\infty f(t)e^{-st} \, dt.
\]

The Laplace integrator is \( dx = e^{-st} \, dt \) instead of the usual \( dt \).

A Laplace integral is succinctly denoted in science and engineering literature by the symbol

\[
L(f(t)),
\]

which abbreviates

\[
\int_E (f(t)) \, dx,
\]

with set \( E = [0, \infty) \) and Laplace integrator \( dx = e^{-st} \, dt \).
A First LaPlace Table

\[
\int_0^\infty (1)e^{-st}\,dt = -\left(\frac{1}{s}\right)e^{-st}\bigg|_{t=0}^{t=\infty} = 1/s
\]

Laplace integral of \(g(t) = 1\).
Assumed \(s > 0\).

\[
\int_0^\infty (t)e^{-st}\,dt = \int_0^\infty -\frac{d}{ds}(e^{-st})\,dt
= -\frac{d}{ds} \int_0^\infty (1)e^{-st}\,dt
= -\frac{d}{ds} (1/s)
= 1/s^2
\]
Laplace integral of \(g(t) = t\).
Use \(L(1) = 1/s\).
Differentiate.

\[
\int_0^\infty (t^2)e^{-st}\,dt = \int_0^\infty -\frac{d}{ds}(te^{-st})\,dt
= -\frac{d}{ds} \int_0^\infty (t)e^{-st}\,dt
= -\frac{d}{ds} (1/s^2)
= 2/s^3
\]
Laplace integral of \(g(t) = t^2\).
Use \(L(t) = 1/s^2\).
Summary

Table 1. Laplace integral $\int_0^\infty g(t)e^{-st}\,dt$ for $g(t) = 1, t$ and $t^2$.

<table>
<thead>
<tr>
<th>Function</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int_0^\infty (1)e^{-st},dt$</td>
<td>$\frac{1}{s}$</td>
</tr>
</tbody>
</table>

In summary, $L(t^n) = \frac{n!}{s^{1+n}}$
Solving differential equations by Laplace methods requires keeping a smallest table of Laplace integrals available, usually memorized. The last three entries will be verified later.

Table 2. A minimal Laplace integral table with $L$-notation

<table>
<thead>
<tr>
<th>Integral</th>
<th>Laplace Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int_0^\infty (t^n)e^{-st} , dt = \frac{n!}{s^{1+n}}$</td>
<td>$L(t^n) = \frac{n!}{s^{1+n}}$</td>
</tr>
<tr>
<td>$\int_0^\infty (e^{at})e^{-st} , dt = \frac{1}{s-a}$</td>
<td>$L(e^{at}) = \frac{1}{s-a}$</td>
</tr>
<tr>
<td>$\int_0^\infty (\cos bt)e^{-st} , dt = \frac{s}{s^2 + b^2}$</td>
<td>$L(\cos bt) = \frac{s}{s^2 + b^2}$</td>
</tr>
<tr>
<td>$\int_0^\infty (\sin bt)e^{-st} , dt = \frac{b}{s^2 + b^2}$</td>
<td>$L(\sin bt) = \frac{b}{s^2 + b^2}$</td>
</tr>
</tbody>
</table>
The forward table finds the Laplace integral $L(f(t))$ when $f(t)$ is a linear combinations of atoms. The Laplace calculus rules apply to find the Laplace integral of $f(t)$ when it is not in this short table.

### Table 3. Forward Laplace integral table

<table>
<thead>
<tr>
<th>Function $f(t)$</th>
<th>Laplace Integral $L(f(t))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>$t^n$</td>
<td>$\frac{n!}{s^{1+n}}$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$</td>
</tr>
<tr>
<td>$\cos bt$</td>
<td>$\frac{s}{s^2 + b^2}$</td>
</tr>
<tr>
<td>$\sin bt$</td>
<td>$\frac{b}{s^2 + b^2}$</td>
</tr>
</tbody>
</table>
Backward Laplace Table

The backward table finds $f(t)$ from a Laplace integral $L(f(t))$ expression. Always, $f(t)$ is a linear combinations of atoms. The Laplace calculus rules apply to find $f(t)$ when it is does not appear in this short table.

Table 4. Backward Laplace integral table

<table>
<thead>
<tr>
<th>Laplace Integral $L(f(t))$</th>
<th>$f(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{s}$</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{1}{s^{1+n}}$</td>
<td>$t^n$</td>
</tr>
<tr>
<td>$\frac{1}{s-a}$</td>
<td>$e^{at}$</td>
</tr>
<tr>
<td>$\frac{s}{s^2 + b^2}$</td>
<td>$\cos bt$</td>
</tr>
<tr>
<td>$\frac{1}{s^2 + b^2}$</td>
<td>$\sin bt$</td>
</tr>
<tr>
<td></td>
<td>$\frac{b}{b}$</td>
</tr>
</tbody>
</table>
Some Transform Rules

\[ L(f(t) + g(t)) = L(f(t)) + L(g(t)) \]

The integral of a sum is the sum of the integrals.

\[ L(cf(t)) = cL(f(t)) \]

Constants \( c \) pass through the integral sign.

\[ L(y'(t)) = sL(y(t)) - y(0) \]

The \( t \)-derivative rule, or integration by parts.
Lerch’s Cancelation Law and the Fundamental Theorem of Calculus

\[ L(y(t)) = L(f(t)) \implies y(t) = f(t) \quad \text{Lerch’s cancelation law.} \]

Lerch’s cancelation law in integral form is

\[ \int_0^\infty y(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt \quad \text{implies} \quad y(t) = f(t). \]  

Quadrature Methods

Lerch’s Theorem is used last in Laplace’s quadrature method. In Newton calculus, the quadrature method uses the Fundamental Theorem of Calculus first. The two theorems have a similar use, to isolate the solution \( y \) of the differential equation.
Laplace’s method will be applied to solve the initial value problem

\[ y' = -1, \quad y(0) = 0. \]
Table 5. Laplace method details for $y' = -1, y(0) = 0$.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y'(t)e^{-st}dt = -e^{-st}dt$</td>
<td>Multiply $y' = -1$ by $e^{-st}dt$.</td>
</tr>
<tr>
<td>$\int_0^\infty y'(t)e^{-st}dt = \int_0^\infty -e^{-st}dt$</td>
<td>Integrate $t = 0$ to $t = \infty$.</td>
</tr>
<tr>
<td>$\int_0^\infty y'(t)e^{-st}dt = -1/s$</td>
<td>Use Table 1.</td>
</tr>
<tr>
<td>$s \int_0^\infty y(t)e^{-st}dt - y(0) = -1/s$</td>
<td>Integrate by parts on the left.</td>
</tr>
<tr>
<td>$\int_0^\infty y(t)e^{-st}dt = -1/s^2$</td>
<td>Use $y(0) = 0$ and divide.</td>
</tr>
<tr>
<td>$\int_0^\infty y(t)e^{-st}dt = \int_0^\infty (-t)e^{-st}dt$</td>
<td>Use Table 1.</td>
</tr>
<tr>
<td>$y(t) = -t$</td>
<td>Apply Lerch’s cancellation law.</td>
</tr>
</tbody>
</table>
### Translation to $L$-notation

**Table 6.** Laplace method $L$-notation details for $y' = -1$, $y(0) = 0$ translated from Table 5.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L(y'(t)) = L(-1)$</td>
<td>Apply $L$ across $y' = -1$, or multiply $y' = -1$ by $e^{-st}dt$, integrate $t = 0$ to $t = \infty$.</td>
</tr>
<tr>
<td>$L(y'(t)) = -1/s$</td>
<td>Use Table 1 forwards.</td>
</tr>
<tr>
<td>$sL(y(t)) - y(0) = -1/s$</td>
<td>Integrate by parts on the left.</td>
</tr>
<tr>
<td>$L(y(t)) = -1/s^2$</td>
<td>Use $y(0) = 0$ and divide.</td>
</tr>
<tr>
<td>$L(y(t)) = L(-t)$</td>
<td>Apply Table 1 backwards.</td>
</tr>
<tr>
<td>$y(t) = -t$</td>
<td>Invoke Lerch’s cancelation law.</td>
</tr>
</tbody>
</table>
Example (Laplace method) Solve by Laplace’s method the initial value problem \( y' = 5 - 2t, \ y(0) = 1 \) to obtain \( y(t) = 1 + 5t - t^2 \).

Solution: Laplace’s method is outlined in Tables 5 and 6. The \( L \)-notation of Table 6 will be used to find the solution \( y(t) = 1 + 5t - t^2 \).

\[
L(y'(t)) = L(5 - 2t) \\
= 5L(1) - 2L(t) \\
= \frac{5}{s} - \frac{2}{s^2}
\]

Apply \( L \) across \( y' = 5 - 2t \). Linearity of the transform.

\[
sL(y(t)) - y(0) = \frac{5}{s} - \frac{2}{s^2}
\]

Use Table 1 forwards.

\[
L(y(t)) = \frac{1}{s} + \frac{5}{s^2} - \frac{2}{s^3}
\]

Apply the \( t \)-derivative rule. Use \( y(0) = 1 \) and divide.

\[
L(y(t)) = L(1) + 5L(t) - L(t^2)
\]

Use Table 1 backwards. Linearity of the transform.

\[
y(t) = 1 + 5t - t^2
\]

Invoke Lerch’s cancelation law.
2 Example (Laplace method) Solve by Laplace’s method the initial value problem
\( y'' = 10, \ y(0) = y'(0) = 0 \) to obtain \( y(t) = 5t^2 \).

**Solution:** The \( L \)-notation of Table 6 will be used to find the solution \( y(t) = 5t^2 \).

\[
\begin{align*}
L(y''(t)) &= L(10) \\
sL(y'(t)) - y'(0) &= L(10) \\
s'[sL(y(t)) - y(0)] - y'(0) &= L(10) \\
s^2L(y(t)) &= 10L(1) \\
L(y(t)) &= \frac{10}{s^3} \\
L(y(t)) &= L(5t^2) \\
y(t) &= 5t^2
\end{align*}
\]

- Apply \( L \) across \( y'' = 10 \).
- Apply the \( t \)-derivative rule to \( y' \).
- Repeat the \( t \)-derivative rule, on \( y \).
- Use \( y(0) = y'(0) = 0 \).
- Use Table 1 forwards. Then divide.
- Use Table 1 backwards.
- Invoke Lerch’s cancelation law.