An RREF Method for Finding Inverses

An efficient method to find the inverse $B$ of a square matrix $A$, should it happen to exist, is to form the augmented matrix $C = \text{aug}(A, I)$ and then read off $B$ as the package of the last $n$ columns of $\text{rref}(C)$. This method is based upon the equivalence

$$\text{rref}(\text{aug}(A, I)) = \text{aug}(I, B) \quad \text{if and only if} \quad AB = I.$$
Main Results

Theorem 1 (Inverse Test)
If \( A \) and \( B \) are square matrices such that \( AB = I \), then also \( BA = I \). Therefore, only one of the equalities \( AB = I \) or \( BA = I \) is required to check an inverse.

Theorem 2 (The \texttt{rref} Inversion Method)
Let \( A \) and \( B \) denote square matrices. Then

(a) If \( \text{rref}(\text{aug}(A, I)) = \text{aug}(I, B) \), then \( AB = BA = I \) and \( B \) is the inverse of \( A \).

(b) If \( AB = BA = I \), then \( \text{rref}(\text{aug}(A, I)) = \text{aug}(I, B) \).

(c) If \( \text{rref}(\text{aug}(A, I)) = \text{aug}(C, B) \) and \( C \neq I \), then \( A \) is not invertible.
Finding inverses

The \textbf{rref} inversion method will be illustrated for the matrix

\[
C = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
0 & 1 & 1
\end{pmatrix}.
\]

Define the first frame of the sequence to be \(C_1 = \text{aug}(C, I)\), then compute the frame sequence to \text{rref}(C) as follows.
\[ C_1 = \begin{pmatrix} 1 & 0 & 1 | & 1 & 0 & 0 \\ 0 & 1 & -1 | & 0 & 1 & 0 \\ 0 & 1 & 1 | & 0 & 0 & 1 \end{pmatrix} \]  
First Frame

\[ C_2 = \begin{pmatrix} 1 & 0 & 1 | & 1 & 0 & 0 \\ 0 & 1 & -1 | & 0 & 1 & 0 \\ 0 & 0 & 2 | & 0 & -1 & 1 \end{pmatrix} \]  
combo(3, 2, -1)

\[ C_3 = \begin{pmatrix} 1 & 0 & 1 | & 1 & 0 & 0 \\ 0 & 1 & -1 | & 0 & 1 & 0 \\ 0 & 0 & 1 | & 0 & -1/2 & 1/2 \end{pmatrix} \]  
mult(3, 1/2)

\[ C_4 = \begin{pmatrix} 1 & 0 & 1 | & 1 & 0 & 0 \\ 0 & 1 & 0 | & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 | & 0 & -1/2 & 1/2 \end{pmatrix} \]  
combo(3, 2, 1)

\[ C_5 = \begin{pmatrix} 1 & 0 & 0 | & 1 & 1/2 & -1/2 \\ 0 & 1 & 0 | & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 | & 0 & -1/2 & 1/2 \end{pmatrix} \]  
combo(3, 1, -1)

Last Frame
The theory

$$rref(\text{aug}(A, I)) = \text{aug}(I, B) \quad \text{if and only if} \quad AB = I$$

implies that the inverse of $A$ is the matrix in the right panel of the last frame

$$C_5 = \begin{pmatrix}
1 & 0 & 0 & 1 & 1/2 & -1/2 \\
0 & 1 & 0 & 0 & 1/2 & 1/2 \\
0 & 0 & 1 & 0 & -1/2 & 1/2
\end{pmatrix}.$$

Then

$$A^{-1} = \begin{pmatrix}
1 & 1/2 & -1/2 \\
0 & 1/2 & 1/2 \\
0 & -1/2 & 1/2
\end{pmatrix}.$$