Geometry of linear transformations

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Description</th>
</tr>
</thead>
</table>
| \(
\begin{pmatrix}
  k & 0 \\
  0 & k
\end{pmatrix}
\) | Scaling |
| \(\begin{pmatrix}
  1 & 0 \\
  0 & 0
\end{pmatrix}\) | Projection |
| \(\begin{pmatrix}
  -1 & 0 \\
  0 & 1
\end{pmatrix}\) | Reflection |
| \(\begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix}\) | Rotation |
| \(\begin{pmatrix}
  0 & 2 \\
  -2 & 0
\end{pmatrix}\) | Scaled Rotation |
| \(\begin{pmatrix}
  1 & 0 \\
  k & 1
\end{pmatrix}\) | Vertical Shear |
| \(\begin{pmatrix}
  1 & k \\
  0 & 1
\end{pmatrix}\) | Horizontal Shear |

Sub-classes: **Dilation** \((k > 1)\) and **Contraction** \((0 < k < 1)\).

Projection: Define \(\text{proj}_L(x) = (x \cdot u)u\) where \(u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\) is a unit vector, \(u_1^2 + u_2^2 = 1\). The matrix is \(\begin{pmatrix} u_1 u_1 & u_1 u_2 \\ u_1 u_2 & u_2 u_2 \end{pmatrix}\).

Reflection: Define \(\text{refl}_L(x) = 2(x \cdot u)u - x\). The matrix is \(\begin{pmatrix} a & b \\ b & -a \end{pmatrix}\), \(a^2 + b^2 = 1\).

Rotation: In general, \(\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}\).

Scaled Rotation: In general, \(\begin{pmatrix} r \cos \theta & r \sin \theta \\ -r \sin \theta & r \cos \theta \end{pmatrix}\).

Vertical Shear: Change vertical \(y \rightarrow y + kx\), leave \(x\) fixed.

Horizontal Shear: Change horizontal \(x \rightarrow x + ky\), leave \(y\) fixed.
Properties of Geometric Transformations

- The columns of a projection matrix are scalar multiples of a single unit vector $\mathbf{u}$, therefore the columns are either the zero vector or else a vector parallel to $\mathbf{u}$.
- The columns of a reflection matrix are unit vectors that are pairwise orthogonal, that is, their pairwise dot products are zero.
- A shear can be classified as horizontal or vertical by its effect in mapping columns of the identity matrix. A horizontal shear leaves the first column of the identity matrix fixed, whereas a vertical shear leaves the second column of the identity matrix fixed.