Determinant Theory

- Unique Solution of $Ax = b$
- College Algebra Definition of Determinant
- Diagram for Sarrus’ $3 \times 3$ Rule
- Transpose Rule
- How to Compute the Value of any Determinant
  - Four Rules to Compute any Determinant
  - Special Determinant Rules
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- Adjugate Formula for the Inverse
- Determinant Product Theorem
Unique Solution of a $2 \times 2$ System

The $2 \times 2$ system

\begin{align*}
ax + by &= e, \\
 cx + dy &= f, \\
\end{align*}

(1)

has a unique solution provided $\Delta = ad - bc$ is nonzero, in which case the solution is given by

\begin{align*}
x &= \frac{de - bf}{ad - bc}, \\
y &= \frac{af - ce}{ad - bc}.
\end{align*}

(2)

This result is called Cramer’s Rule for $2 \times 2$ systems, learned in college algebra.
Determinants of Order 2

College algebra introduces matrix notation and determinant notation:

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{det}(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}. \]

Evaluation of \( \text{det}(A) \) is by **Sarrus’ 2 \times 2 Rule**:

\[
\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.
\]

The first product \( ad \) is the product of the main diagonal entries and the other product \( bc \) is from the anti-diagonal.

Cramer’s 2 \times 2 rule in determinant notation is

\[
x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.
\]
Relation to Inverse Matrices

System
(4) \[ \begin{align*}
    ax + by &= e, \\
    cx + dy &= f,
\end{align*} \]
can be expressed as the vector-matrix system \( Au = b \) where
\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad b = \begin{pmatrix} e \\ f \end{pmatrix}.
\]

Inverse matrix theory implies
\[
A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad u = A^{-1}b = \frac{1}{ad - bc} \begin{pmatrix} de - bf \\ af - ce \end{pmatrix}.
\]

Cramer’s Rule is a compact summary of the unique solution of system (4).
Unique Solution of an $n \times n$ System

System

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
  \vdots & \quad \vdots \quad \ddots \quad \vdots \\
  a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{align*}
\]

(5)

can be written as an $n \times n$ vector-matrix equation $A\vec{x} = \vec{b}$, where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{b} = (b_1, \ldots, b_n)$. The system has a unique solution provided the determinant of coefficients $\Delta = \det(A)$ is nonzero, and then Cramer’s Rule for $n \times n$ systems gives

\[
\begin{align*}
  x_1 &= \frac{\Delta_1}{\Delta}, \\
  x_2 &= \frac{\Delta_2}{\Delta}, \quad \ldots, \\
  x_n &= \frac{\Delta_n}{\Delta}.
\end{align*}
\]

(6)

Symbol $\Delta_j = \det(B)$, where matrix $B$ has the same columns as matrix $A$, except $\text{col}(B, j) = \vec{b}$. 
Determinants of Order $n$

Determinants will be defined shortly; intuition from the $2 \times 2$ case and Sarrus’ rule should suffice for the moment.
Determinant Notation for Cramer’s Rule

The determinant of coefficients for system \( A\vec{x} = \vec{b} \) is denoted by

\[
\Delta = \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}.
\]

The other \( n \) determinants in Cramer’s rule (6) are given by

\[
\Delta_1 = \begin{vmatrix}
b_1 & a_{12} & \cdots & a_{1n} \\
b_2 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
b_n & a_{n2} & \cdots & a_{nn}
\end{vmatrix}, \ldots, \Delta_n = \begin{vmatrix}
a_{11} & a_{12} & \cdots & b_1 \\
a_{21} & a_{22} & \cdots & b_2 \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & b_n
\end{vmatrix}.
\]
Given an $n \times n$ matrix $A$, define

$$
\det(A) = \sum_{\sigma \in S_n} (-1)^{\text{parity}(\sigma)} a_{1\sigma_1} \cdots a_{n\sigma_n}.
$$

In the formula, $a_{ij}$ denotes the element in row $i$ and column $j$ of the matrix $A$. The symbol $\sigma = (\sigma_1, \ldots, \sigma_n)$ stands for a rearrangement of the subscripts $1, 2, \ldots, n$ and $S_n$ is the set of all possible rearrangements. The nonnegative integer $\text{parity}(\sigma)$ is determined by counting the minimum number of pairwise interchanges required to assemble the list of integers $\sigma_1, \ldots, \sigma_n$ into natural order $1, \ldots, n$. 
For a $3 \times 3$ matrix, the College Algebra formula reduces to Sarrus’ $3 \times 3$ Rule

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

(10)

$$= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13}.$$
Diagram for Sarrus’ $3 \times 3$ Rule

The number $\det(A)$, in the $3 \times 3$ case, can be computed by the algorithm in Figure 1, which parallels the one for $2 \times 2$ matrices. The $5 \times 3$ array is made by copying the first two rows of $A$ into rows 4 and 5.

**Warning:** there is no Sarrus’ rule diagram for $4 \times 4$ or larger matrices!

**Figure 1.** Sarrus’ rule diagram for $3 \times 3$ matrices, which gives

$$
\det(A) = (a + b + c) - (d + e + f).
$$
**Transpose Rule**

A consequence of the college algebra definition of determinant is the relation

\[
\det(A) = \det(A^T)
\]

where \(A^T\) means the transpose of \(A\), obtained by swapping rows and columns. This relation implies the following.

All determinant theory results for rows also apply to columns.
How to Compute the Value of any Determinant

- **Four Rules.** These are the *Triangular Rule*, *Combination Rule*, *Multiply Rule* and the *Swap Rule*.

- **Special Rules.** These apply to evaluate a determinant as zero.

- **Cofactor Expansion.** This is an iterative scheme which reduces computation of a determinant to a number of smaller determinants.

- **Hybrid Method.** The four rules and the cofactor expansion are combined.
## Four Rules

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
</table>
| Triangular    | The value of \( \det(A) \) for either an upper triangular or a lower triangular matrix \( A \) is the product of the diagonal elements: \[
\det(A) = a_{11}a_{22} \cdots a_{nn}.
\] This is a one-arrow Sarrus’ rule. |
| Swap          | If \( B \) results from \( A \) by swapping two rows, then \[
\det(A) = (-1) \det(B).
\]                                                      |
| Combination   | The value of \( \det(A) \) is unchanged by adding a multiple of a row to a different row. |
| Multiply      | If one row of \( A \) is multiplied by constant \( c \) to create matrix \( B \), then \[
\det(B) = c \det(A).
\]                                                      |
1 Example (Four Properties) Apply the four properties of a determinant to justify the formula

$$\det \begin{pmatrix} 12 & 6 & 0 \\ 11 & 5 & 1 \\ 10 & 2 & 2 \end{pmatrix} = 24.$$
Solution: Let \( D \) denote the value of the determinant. Then

\[
D = \det \begin{pmatrix} 12 & 6 & 0 \\ 11 & 5 & 1 \\ 10 & 2 & 2 \end{pmatrix} \text{ Given.}
\]

\[
= \det \begin{pmatrix} 12 & 6 & 0 \\ -1 & -1 & 1 \\ -2 & -4 & 2 \end{pmatrix} \text{ combo}(1,2,-1), \text{ combo}(1,3,-1). \text{ Combination leaves the determinant unchanged.}
\]

\[
= 6 \det \begin{pmatrix} 2 & 1 & 0 \\ -1 & -1 & 1 \\ -2 & -4 & 2 \end{pmatrix} \text{ Multiply rule } m = 1/6 \text{ on row 1 factors out a 6.}
\]

\[
= 6 \det \begin{pmatrix} 0 & -1 & 2 \\ -1 & -1 & 1 \\ 0 & -3 & 2 \end{pmatrix} \text{ combo}(1,3,1), \text{ combo}(2,1,2).
\]

\[
= -6 \det \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & -3 & 2 \end{pmatrix} \text{ swap}(1,2). \text{ Swap changes the sign of the determinant.}
\]

\[
= 6 \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & -3 & 2 \end{pmatrix} \text{ Multiply rule } m = -1 \text{ on row 1.}
\]

\[
= 6 \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{pmatrix} \text{ combo}(2,3,-3).
\]

\[
= 6(1)(-1)(-4) = 24 \text{ Triangular rule. Formula verified.}
\]
Elementary Matrices and the Four Rules

The four rules can be stated in terms of elementary matrices as follows.

**Triangular**

The value of $\det(A)$ for either an upper triangular or a lower triangular matrix $A$ is the product of the diagonal elements: $\det(A) = a_{11}a_{22}\cdots a_{nn}$. This is a one-arrow Sarrus’ rule valid for dimension $n$.

**Swap**

If $E$ is an elementary matrix for a swap rule, then $\det(EA) = (-1) \det(A)$.

**Combination**

If $E$ is an elementary matrix for a combination rule, then $\det(EA) = \det(A)$.

**Multiply**

If $E$ is an elementary matrix for a multiply rule with multiplier $m \neq 0$, then $\det(EA) = m \det(A)$.

Because $\det(E) = 1$ for a combination rule, $\det(E) = -1$ for a swap rule and $\det(E) = c$ for a multiply rule with multiplier $c \neq 0$, it follows that for any elementary matrix $E$ there is the **determinant multiplication rule**

$$\det(EA) = \det(E) \det(A).$$
### Special Determinant Rules

The results are stated for rows but also hold for columns, because $\det(A) = \det(A^T)$.

<table>
<thead>
<tr>
<th>Rule</th>
<th>Condition</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero row</td>
<td>If one row of $A$ is zero, then $\det(A) = 0$.</td>
<td>$\det(A) = 0$</td>
</tr>
<tr>
<td>Duplicate rows</td>
<td>If two rows of $A$ are identical, then $\det(A) = 0$.</td>
<td>$\det(A) = 0$</td>
</tr>
<tr>
<td>RREF $\neq I$</td>
<td>If $\text{rref}(A) \neq I$, then $\det(A) = 0$.</td>
<td>$\det(A) = 0$</td>
</tr>
<tr>
<td>Common factor</td>
<td>The relation $\det(A) = c \det(B)$ holds, provided $A$ and $B$ differ only in one row, say row $j$, for which $\text{row}(A, j) = c \text{row}(B, j)$.</td>
<td>$\det(A) = c \det(B)$</td>
</tr>
<tr>
<td>Row linearity</td>
<td>The relation $\det(A) = \det(B) + \det(C)$ holds, provided $A$, $B$ and $C$ differ only in one row, say row $j$, for which $\text{row}(A, j) = \text{row}(B, j) + \text{row}(C, j)$.</td>
<td>$\det(A) = \det(B) + \det(C)$</td>
</tr>
</tbody>
</table>
Cofactor Expansion for $3 \times 3$ Matrices

This is a review the college algebra topic, where the dimension of $A$ is $3$. Cofactor row expansion means the following formulas are valid:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(+1) \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(+1) \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{21}(-1) \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}(+1) \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}(-1) \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{31}(+1) \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{32}(-1) \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}(+1) \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

The formulas expand a $3 \times 3$ determinant in terms of $2 \times 2$ determinants, along a row of $A$. The attached signs $\pm 1$ are called the checkerboard signs, to be defined shortly. The $2 \times 2$ determinants are called minors of the $3 \times 3$ determinant $|A|$. The checkerboard sign together with a minor is called a cofactor.
Cofactor Expansion Illustration

Cofactor expansion formulas are generally used when a row has one or two zeros, making it unnecessary to evaluate one or two of the $2 \times 2$ determinants in the expansion. To illustrate, row 1 cofactor expansion gives

\[
\begin{vmatrix}
3 & 0 & 0 \\
2 & 1 & 7 \\
5 & 4 & 8
\end{vmatrix}
= 3(+1) \begin{vmatrix} 1 & 7 \\ 4 & 8 \end{vmatrix} + 0(-1) \begin{vmatrix} 2 & 7 \\ 5 & 8 \end{vmatrix} + 0(+1) \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}
= 3(+1)(8 - 28) + 0 + 0
= -60.
\]

What has been said for rows also applies to columns, due to the transpose formula

\[\det(A) = \det(A^T).\]
Minor

The \((n - 1) \times (n - 1)\) determinant obtained from \(\text{det}(A)\) by striking out row \(i\) and column \(j\) is called the \((i, j)\)–minor of \(A\) and denoted \(\text{minor}(A, i, j)\). Literature might use \(M_{ij}\) for a minor.

Cofactor

The \((i, j)\)–cofactor of \(A\) is \(\text{cof}(A, i, j) = (-1)^{i+j} \text{minor}(A, i, j)\). Multiplicative factor \((-1)^{i+j}\) is called the checkerboard sign, because its value can be determined by counting plus, minus, plus, etc., from location \((1, 1)\) to location \((i, j)\) in any checkerboard fashion.

Expansion of Determinants by Cofactors

\[
\text{det}(A) = \sum_{j=1}^{n} a_{kj} \text{cof}(A, k, j), \quad \text{det}(A) = \sum_{i=1}^{n} a_{i\ell} \text{cof}(A, i, \ell),
\]

In (11), \(1 \leq k \leq n, 1 \leq \ell \leq n\). The first expansion is called a cofactor row expansion and the second is called a cofactor column expansion. The value \(\text{cof}(A, i, j)\) is the cofactor of element \(a_{ij}\) in \(\text{det}(A)\), that is, the checkerboard sign times the minor of \(a_{ij}\).
2 Example (Hybrid Method) Justify by cofactor expansion and the four properties the identity

\[
\begin{vmatrix}
10 & 5 & 0 \\
11 & 5 & a \\
10 & 2 & b \\
\end{vmatrix} = 5(6a - b).
\]

Solution: Let \( D \) denote the value of the determinant. Then

\[
D = \det \begin{vmatrix}
10 & 5 & 0 \\
11 & 5 & a \\
10 & 2 & b \\
\end{vmatrix} \quad \text{Given.}
\]

\[
= \det \begin{vmatrix}
10 & 5 & 0 \\
1 & 0 & a \\
0 & -3 & b \\
\end{vmatrix} \quad \text{Combination leaves the determinant unchanged: combo(1,2,-1), combo(1,3,-1).}
\]

\[
= \det \begin{vmatrix}
0 & 5 & -10a \\
1 & 0 & a \\
0 & -3 & b \\
\end{vmatrix} \quad \text{combo(2,1,-10).}
\]

\[
= (1)(-1) \det \begin{vmatrix}
5 & -10a \\
-3 & b \\
\end{vmatrix} \quad \text{Cofactor expansion on column 1.}
\]

\[
= (1)(-1)(5b - 30a) \quad \text{Sarrus’ rule for } n = 2.
\]

\[
= 5(6a - b). \quad \text{Formula verified.}
\]
3 Example (Cramer’s Rule) Solve by Cramer’s rule the system of equations

\[
\begin{align*}
2x_1 + 3x_2 + x_3 - x_4 &= 1, \\
x_1 + x_2 - x_4 &= -1, \\
3x_2 + x_3 + x_4 &= 3, \\
x_1 + x_3 - x_4 &= 0,
\end{align*}
\]

verifying \( x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 2. \)
Solution: Form the four determinants $\Delta_1, \ldots, \Delta_4$ from the base determinant $\Delta$ as follows:

$$\Delta = \det \begin{pmatrix} 2 & 3 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix},$$

$$\Delta_1 = \det \begin{pmatrix} 1 & 3 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ 3 & 3 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \Delta_2 = \det \begin{pmatrix} 2 & 1 & 1 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix},$$

$$\Delta_3 = \det \begin{pmatrix} 2 & 3 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 3 & 3 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad \Delta_4 = \det \begin{pmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & 3 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Five repetitions of the methods used in the previous examples give the answers $\Delta = -2$, $\Delta_1 = -2$, $\Delta_2 = 0$, $\Delta_3 = -2$, $\Delta_4 = -4$, therefore Cramer's rule implies the solution

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad x_3 = \frac{\Delta_3}{\Delta}, \quad x_4 = \frac{\Delta_4}{\Delta}.$$

Then $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 2$. 
The details of the computation above can be checked in computer algebra system `maple` as follows.

```maple
with(linalg):
A:=matrix([[2, 3, 1, -1], [1, 1, 0, -1], [0, 3, 1, 1], [1, 0, 1, -1]]);
Delta:= det(A);
b:=vector([1,-1,3,0]):
B1:=A: col(B1,1):=b:
Delta1:=det(B1);
x[1]:=Delta1/Delta;
```
The Adjugate Matrix

The adjugate \( \text{adj}(A) \) of an \( n \times n \) matrix \( A \) is the transpose of the matrix of cofactors,

\[
\text{adj}(A) = \begin{pmatrix} \text{cof}(A, 1, 1) & \text{cof}(A, 1, 2) & \cdots & \text{cof}(A, 1, n) \\ \text{cof}(A, 2, 1) & \text{cof}(A, 2, 2) & \cdots & \text{cof}(A, 2, n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cof}(A, n, 1) & \text{cof}(A, n, 2) & \cdots & \text{cof}(A, n, n) \end{pmatrix}^T.
\]

A cofactor \( \text{cof}(A, i, j) \) is the checkerboard sign \( (-1)^{i+j} \) times the corresponding minor determinant \( \text{minor}(A, i, j) \).

Adjugate of a \( 2 \times 2 \)

\[
\text{adj} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}
\]

In words: swap the diagonal elements and change the sign of the off-diagonal elements.
Adjugate Formula for the Inverse

For any $n \times n$ matrix

$$A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = \det(A) I.$$ 

The equation is valid even if $A$ is not invertible. The relation suggests several ways to find $\det(A)$ from $A$ and $\text{adj}(A)$ with one dot product.

For an invertible matrix $A$, the relation implies $A^{-1} = \frac{\text{adj}(A)}{\det(A)}$:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} \text{cof}(A, 1, 1) & \text{cof}(A, 1, 2) & \cdots & \text{cof}(A, 1, n) \\ \text{cof}(A, 2, 1) & \text{cof}(A, 2, 2) & \cdots & \text{cof}(A, 2, n) \\ \vdots & \vdots & \cdots & \vdots \\ \text{cof}(A, n, 1) & \text{cof}(A, n, 2) & \cdots & \text{cof}(A, n, n) \end{pmatrix}^T$$
Application: Adjugate Shortcut

Given \( A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \), then we can compute \( \text{adj}(A) = \begin{pmatrix} 1 & 3 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 3 \end{pmatrix} \).

Suppose that we mark some unknown entries in \( \text{adj}(A) \) by \( ? \) and write \( |A| \) for \( \det(A) \). Then the formula \( A \text{adj}(A) = \text{adj}(A)A = \det(A)I \) becomes

\[
\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} ? & 3 & ? \\ ? & 1 & ? \\ ? & -1 & ? \end{pmatrix} = \begin{pmatrix} ? & 3 & ? \\ ? & 1 & ? \\ ? & -1 & ? \end{pmatrix} \begin{pmatrix} 1 & 3 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}.
\]

While the second product \( \text{adj}(A)A \) contains useless information, the first product gives \( \text{row}(A, 2) \text{col}(\text{adj}(A), 2) = \det(A) \). Because the values are known, then \( \det(A) = 6 + 1 + 0 = 7 \).

Knowing \( A \) and \( \text{adj}(A) \) gives the value of \( \det(A) \) in one dot product.
Elementary Matrices

**Theorem 1 (Determinants and Elementary Matrices)**

Let $E$ be an $n \times n$ elementary matrix. Then

- **Combination** $\det(E) = 1$
- **Multiply** $\det(E) = m$ for multiplier $m$.
- **Swap** $\det(E) = -1$
- **Product** $\det(EX) = \det(E) \det(X)$ for all $n \times n$ matrices $X$.

**Theorem 2 (Determinants and Invertible Matrices)**

Let $A$ be a given invertible matrix. Then

$$\det(A) = \frac{(-1)^s}{m_1 m_2 \cdots m_r}$$

where $s$ is the number of swap rules applied and $m_1, m_2, \ldots, m_r$ are the nonzero multipliers used in multiply rules when $A$ is reduced to $\text{rref}(A)$. 
Theorem 3 (Determinant Product Rule)
Let $A$ and $B$ be given $n \times n$ matrices. Then

$$\det(AB) = \det(A) \det(B).$$

Proof

Assume $A^{-1}$ does not exist. Then $A$ has zero determinant, which implies $\det(A) \det(B) = 0$. If $\det(B) = 0$, then $Bx = 0$ has infinitely many solutions, in particular a nonzero solution $x$. Multiply $Bx = 0$ by $A$, then $ABx = 0$ which implies $AB$ is not invertible. Then the identity $\det(AB) = \det(A) \det(B)$ holds, because both sides are zero. If $\det(B) \neq 0$ but $\det(A) = 0$, then there is a nonzero $y$ with $Ay = 0$. Define $x = B^{-1}y$. Then $ABx = Ay = 0$, with $x \neq 0$, which implies $AB$ is not invertible, and as earlier in this paragraph, the identity holds. This completes the proof when $A$ is not invertible.

Assume $A$ is invertible. In particular, $\text{rref}(A^{-1}) = I$. Write $I = \text{rref}(A^{-1}) = E_1E_2\cdots E_kA^{-1}$ for elementary matrices $E_1, \ldots, E_k$. Then $A = E_1E_2\cdots E_k$ and

$$AB = E_1E_2\cdots E_kB.$$

The theorem follows from repeated application of the basic identity $\det(EX) = \det(E) \det(X)$ to relation (12), because

$$\det(AB) = \det(E_1) \cdots \det(E_k) \det(B) = \det(A) \det(B).$$