Systems of Differential Equations
Matrix Methods

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Characteristic Equation

**Definition 1 (Characteristic Equation)**

Given a square matrix $A$, the **characteristic equation** of $A$ is the polynomial equation

$$\det(A - rI) = 0.$$ 

The determinant $\det(A - rI)$ is formed by subtracting $r$ from the diagonal of $A$. The polynomial $p(r) = \det(A - rI)$ is called the **characteristic polynomial**.

- If $A$ is $2 \times 2$, then $p(r)$ is a quadratic.
- If $A$ is $3 \times 3$, then $p(r)$ is a cubic.
- The determinant is expanded by the cofactor rule, in order to preserve factorizations.
Characteristic Equation Examples

Create $\det(A - rI)$ by subtracting $r$ from the diagonal of $A$.

Evaluate by the cofactor rule.

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}, \quad p(r) = \begin{vmatrix} 2 - r & 3 \\ 0 & 4 - r \end{vmatrix} = (2 - r)(4 - r)$$

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}, \quad p(r) = \begin{vmatrix} 2 - r & 3 & 4 \\ 0 & 5 - r & 6 \\ 0 & 0 & 7 - r \end{vmatrix} = (2-r)(5-r)(7-r)$$
**Cayley-Hamilton Theorem 1 (Cayley-Hamilton)**

A square matrix $A$ satisfies its own characteristic equation.

If $p(r) = (-r)^n + a_{n-1}(-r)^{n-1} + \cdots + a_0$, then the result is the equation

$$(-A)^n + a_{n-1}(-A)^{n-1} + \cdots + a_1(-A) + a_0I = 0,$$

where $I$ is the $n \times n$ identity matrix and $0$ is the $n \times n$ zero matrix.

**The 2 × 2 Case**

Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and for $a_1 = \text{trace}(A)$, $a_0 = \det(A)$ we have $p(r) = r^2 + a_1(-r) + a_0$. The Cayley-Hamilton theorem says

$$A^2 + a_1(-A) + a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$
Cayley-Hamilton Example

Assume

\[ A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix} \]

Then

\[ p(r) = \begin{vmatrix} 2 - r & 3 & 4 \\ 0 & 5 - r & 6 \\ 0 & 0 & 7 - r \end{vmatrix} = (2 - r)(5 - r)(7 - r) \]

and the Cayley-Hamilton Theorem says that

\[ (2I - A)(5I - A)(7I - A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]
Theorem 2 (Cayley-Hamilton-Ziebur Structure Theorem for \( \vec{u}' = A\vec{u} \))

A component function \( u_k(t) \) of the vector solution \( \vec{u}(t) \) for \( \vec{u}'(t) = A\vec{u}(t) \) is a solution of the \( n \)th order linear homogeneous constant-coefficient differential equation whose characteristic equation is \( \det(A - rI) = 0 \).

The theorem implies that the vector solution \( \vec{u}(t) \) of

\[
\vec{u}' = A\vec{u}
\]

is a vector linear combination of atoms constructed from the roots of the characteristic equation \( \det(A - rI) = 0 \).
Proof of the Cayley-Hamilton-Ziebur Theorem

Consider the case \( n = 2 \), because the proof details are similar in higher dimensions.

\[
r^2 + a_1 r + a_0 = 0 \quad \text{Expanded characteristic equation}
\]
\[
A^2 + a_1 A + a_0 I = 0 \quad \text{Cayley-Hamilton matrix equation}
\]
\[
A^2 \vec{u} + a_1 A \vec{u} + a_0 \vec{u} = \vec{0} \quad \text{Right-multiply by} \quad \vec{u} = \vec{u}(t)
\]
\[
\vec{u}'' = A \vec{u}' = A^2 \vec{u} \quad \text{Differentiate} \quad \vec{u}' = A \vec{u}
\]
\[
\vec{u}'' + a_1 \vec{u}' + a_0 \vec{u} = \vec{0} \quad \text{Replace} \quad A^2 \vec{u} \rightarrow \vec{u}'', \ A \vec{u} \rightarrow \vec{u}'
\]

Then the components \( x(t), y(t) \) of \( \vec{u}(t) \) satisfy the two differential equations

\[
x''(t) + a_1 x'(t) + a_0 x(t) = 0,
\]
\[
y''(t) + a_1 y'(t) + a_0 y(t) = 0.
\]

This system implies that the components of \( \vec{u}(t) \) are solutions of the second order DE with characteristic equation \( \det(A - rI) = 0 \).
Cayley-Hamilton-Ziebur Method

The Cayley-Hamilton-Ziebur Method for $\vec{u}' = A\vec{u}$

Let $\text{atom}_1, \ldots, \text{atom}_n$ denote the atoms constructed from the $n$th order characteristic equation $\det(A - rI) = 0$ by Euler’s Theorem. The solution of

$$\vec{u}' = A\vec{u}$$

is given for some constant vectors $\vec{d}_1, \ldots, \vec{d}_n$ by the equation

$$\vec{u}(t) = (\text{atom}_1)\vec{d}_1 + \cdots + (\text{atom}_n)\vec{d}_n$$
Cayley-Hamilton-Ziebur Method Conclusions

• Solving $\vec{u}' = A\vec{u}$ is reduced to finding the constant vectors $\vec{d}_1, \ldots, \vec{d}_n$.

• The vectors $\vec{d}_j$ are not arbitrary. They are uniquely determined by $A$ and $\vec{u}(0)$!

  A general method to find them is to differentiate the equation

  $$\vec{u}(t) = (\text{atom}_1)\vec{d}_1 + \cdots + (\text{atom}_n)\vec{d}_n$$

  $n - 1$ times, then set $t = 0$ and replace $\vec{u}^{(k)}(0)$ by $A^k\vec{u}(0)$ [because $\vec{u}' = A\vec{u}$, $\vec{u}'' = A\vec{u}' = AA\vec{u}$, etc]. The resulting $n$ equations in vector unknowns $\vec{d}_1, \ldots, \vec{d}_n$ can be solved by elimination.

• If all atoms constructed are base atoms constructed from real roots, then each $\vec{d}_j$ is a constant multiple of a real eigenvector of $A$. Atom $e^{rt}$ corresponds to the eigenpair equation $Av = rv$. 
A 2 × 2 Illustration

Let’s solve $\vec{u}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{u}$, $\vec{u}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

The characteristic polynomial of the non-triangular matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is

$$\begin{vmatrix} 1 - r & 2 \\ 2 & 1 - r \end{vmatrix} = (1 - r)^2 - 4 = (r + 1)(r - 3).$$

Euler’s theorem implies solution atoms are $e^{-t}$, $e^{3t}$.

Then $\vec{u}$ is a vector linear combination of the solution atoms,

$$\vec{u} = e^{-t}\vec{d}_1 + e^{3t}\vec{d}_2.$$
How to Find $\vec{d}_1$ and $\vec{d}_2$ 

We solve for vectors $\vec{d}_1$, $\vec{d}_2$ in the equation

$$\vec{u} = e^{-t}\vec{d}_1 + e^{3t}\vec{d}_2.$$ 

**Advice:** Define $\vec{d}_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Differentiate the above relation. Replace $\vec{u}'$ via $\vec{u}' = A\vec{u}$, then set $t = 0$ and replace $\vec{u}(0)$ by $\vec{d}_0$ in the two formulas to obtain the relations

$$\vec{d}_0 = e^{0}\vec{d}_1 + e^{0}\vec{d}_2$$

$$A\vec{d}_0 = -e^{0}\vec{d}_1 + 3e^{0}\vec{d}_2$$

We solve for $\vec{d}_1$, $\vec{d}_2$ by elimination. Adding the equations gives $\vec{d}_0 + A\vec{d}_0 = 4\vec{d}_2$ and then $\vec{d}_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ implies

$$\vec{d}_1 = \frac{3}{4}\vec{d}_0 - \frac{1}{4}A\vec{d}_0 = \begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix},$$

$$\vec{d}_2 = \frac{1}{4}\vec{d}_0 + \frac{1}{4}A\vec{d}_0 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$
Summary of the $2 \times 2$ Illustration

The solution of the dynamical system

$$\vec{u}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{u}, \quad \vec{u}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

is a vector linear combination of solution atoms $e^{-t}, e^{3t}$ given by the equation

$$\vec{u} = e^{-t} \begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix} + e^{3t} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$ 

Eigenpairs for Free

Each vector appearing in the formula is a scalar multiple of an eigenvector, because eigenvalues $-1, 3$ are real and distinct. The simplified eigenpairs are

$$(-1, \begin{pmatrix} -1 \\ 1 \end{pmatrix}), \quad (3, \begin{pmatrix} 1 \\ 1 \end{pmatrix}).$$
A Matrix Method for Finding $\vec{d}_1$ and $\vec{d}_2$

The Cayley-Hamilton-Ziebur Method produces a unique solution for $\vec{d}_1$, $\vec{d}_2$ because the coefficient matrix

$$
\begin{pmatrix}
  e^0 & e^0 \\
  -e^0 & 3e^0 
\end{pmatrix}
$$

is exactly the Wronskian $W$ of the basis of atoms $e^{-t}, e^{3t}$ evaluated at $t = 0$. This same fact applies no matter the number of coefficients $\vec{d}_1$, $\vec{d}_2$, … to be determined.

The answer for $\vec{d}_1$ and $\vec{d}_2$ can be written in matrix form in terms of the transpose $W^T$ of the Wronskian matrix as

$$
\text{aug}(\vec{d}_1, \vec{d}_2) = \text{aug}(\vec{d}_0, A\vec{d}_0)(W^T)^{-1}.
$$
Solving a $2 \times 2$ Initial Value Problem by the Matrix Method

\[ \vec{u}' = A\vec{u}, \quad \vec{u}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}. \]

Then \( \vec{d}_0 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad A\vec{d}_0 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \) and

\[ \text{aug}(\vec{d}_1, \vec{d}_2) = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} \left( \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}^T \right)^{-1} \begin{pmatrix} 3/2 & 1 \\ 0 & 1/2 \end{pmatrix}. \]

The solution of the initial value problem is

\[ \vec{u}(t) = e^{-t} \begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix} + e^{3t} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -3/2 e^{-t} + 1/2 e^{3t} \\ 3/2 e^{-t} + 1/2 e^{3t} \end{pmatrix}. \]
Other Representations of the Solution $\vec{u}$

Let $y_1(t), \ldots, y_n(t)$ be a solution basis for the $n$th order linear homogeneous constant-coefficient differential equation whose characteristic equation is $\text{det}(A - rI) = 0$.

Consider the solution basis $\text{atom}_1, \text{atom}_2, \ldots, \text{atom}_n$. Each atom is a linear combination of $y_1, \ldots, y_n$. Replacing the atoms in the formula

$$\vec{u}(t) = (\text{atom}_1)\vec{d}_1 + \cdots + (\text{atom}_n)\vec{d}_n$$

by these linear combinations implies there are constant vectors $\vec{D}_1, \ldots, \vec{D}_n$ such that

$$\vec{u}(t) = y_1(t)\vec{D}_1 + \cdots + y_n(t)\vec{D}_n$$
Another General Solution of $\vec{u}' = A\vec{u}$

**Theorem 3 (General Solution)**
The unique solution of $\vec{u}' = Au$, $\vec{u}(0) = \vec{d}_0$ is

$$u(t) = \phi_1(t)u_0 + \phi_2(t)Au_0 + \cdots + \phi_n(t)A^{n-1}u_0$$

where $\phi_1, \ldots, \phi_n$ are linear combinations of atoms constructed from roots of the characteristic equation $\det(A - rI) = 0$, such that

$$\text{Wronskian}(\phi_1(t), \ldots, \phi_n(t))|_{t=0} = I.$$
Proof of the theorem

**Proof**: Details will be given for $n = 3$. The details for arbitrary matrix dimension $n$ is a routine modification of this proof. The Wronskian condition implies $\phi_1$, $\phi_2$, $\phi_3$ are independent. Then each atom constructed from the characteristic equation is a linear combination of $\phi_1$, $\phi_2$, $\phi_3$. It follows that the unique solution $\vec{u}$ can be written for some vectors $\vec{d}_1$, $\vec{d}_2$, $\vec{d}_3$ as

$$\vec{u}(t) = \phi_1(t)\vec{d}_1 + \phi_2(t)\vec{d}_2 + \phi_3(t)\vec{d}_3.$$  

Differentiate this equation twice and then set $t = 0$ in all 3 equations. The relations $\vec{u}' = A\vec{u}$ and $\vec{u}'' = A\vec{u}' = AA\vec{u}$ imply the 3 equations

$$\begin{align*}
\vec{d}_0 &= \phi_1(0)\vec{d}_1 + \phi_2(0)\vec{d}_2 + \phi_3(0)\vec{d}_3 \\
A\vec{d}_0 &= \phi_1'(0)\vec{d}_1 + \phi_2'(0)\vec{d}_2 + \phi_3'(0)\vec{d}_3 \\
A^2\vec{d}_0 &= \phi_1''(0)\vec{d}_1 + \phi_2''(0)\vec{d}_2 + \phi_3''(0)\vec{d}_3
\end{align*}$$

Because the Wronskian is the identity matrix $I$, then these equations reduce to

$$\begin{align*}
\vec{d}_0 &= 1\vec{d}_1 + 0\vec{d}_2 + 0\vec{d}_3 \\
A\vec{d}_0 &= 0\vec{d}_1 + 1\vec{d}_2 + 0\vec{d}_3 \\
A^2\vec{d}_0 &= 0\vec{d}_1 + 0\vec{d}_2 + 1\vec{d}_3
\end{align*}$$

which implies $\vec{d}_1 = \vec{d}_0$, $\vec{d}_2 = A\vec{d}_0$, $\vec{d}_3 = A^2\vec{d}_0$.

The claimed formula for $\vec{u}(t)$ is established and the proof is complete.
Illustrated here is the change of basis formula for $n = 3$. The formula for general $n$ is similar.

Let $\phi_1(t), \phi_2(t), \phi_3(t)$ denote the linear combinations of atoms obtained from the vector formula

$$ (\phi_1(t), \phi_2(t), \phi_3(t)) = (\text{atom}_1(t), \text{atom}_2(t), \text{atom}_3(t)) \ C^{-1} $$

where

$$ C = \text{Wronskian}(\text{atom}_1, \text{atom}_2, \text{atom}_3)(0). $$

The solutions $\phi_1(t), \phi_2(t), \phi_3(t)$ are called the **principal solutions** of the linear homogeneous constant-coefficient differential equation constructed from the characteristic equation $\det(A - rI) = 0$. They satisfy the initial conditions

$$ \text{Wronskian}(\phi_1, \phi_2, \phi_3)(0) = I. $$