Solution Set Basis for Linear Differential Equations

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Linear Differential Equations

The solution set of a homogeneous constant coefficient linear differential equation

\[ y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0y = 0 \]

is known to be a vector space of functions of dimension \( n \), consisting of special linear combinations

(1) \[ y = c_1f_1 + \cdots + c_nf_n, \]

where \( f_1, \ldots, f_n \) are elementary functions known as **atoms**.
Definition of Atom

A base atom is defined to be one of

$$1, e^{ax}, \cos bx, \sin bx, e^{ax} \cos bx, e^{ax} \sin bx,$$

with real $a \neq 0$, $b > 0$.

An atom equals a base atom multiplied by $x^n$, where $n = 0, 1, 2 \ldots$ is an integer.

An atom has coefficient 1, and the zero function is not an atom.

Examples of Atoms

$$1, x, x^2, e^x, xe^{-x}, x^{15} e^{2x} \cos 3x, \cos 3x, \sin 2x, x^2 \cos 2x, x^6 \sin \pi x, x^{10} e^{\pi x} \sin 0.1x$$

Functions that are not Atoms

$$x/(1 + x), \ln |x|, e^{x^2}, \sin(x + 1), 0, 2x, \sin(1/x), \sqrt{x}$$
Theorems about Atoms

**Theorem 1 (Independence)**
Any finite list of atoms is linearly independent.

**Theorem 2 (Euler)**
The real characteristic polynomial $p(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_0$ has a factor $(r - a - ib)^{k+1}$ if and only if

$$x^k e^{ax} \cos bx, \quad x^k e^{ax} \sin bx$$

are real solutions of the differential equation (1). If $b > 0$, then both are atoms. If $b = 0$, then only the first is an atom.

**Theorem 3 (Real Solutions)**
If $u$ and $v$ are real and $u + iv$ is a solution of equation (1), then $u$ and $v$ are real solutions of equation (1).

**Theorem 4 (Basis)**
The solution set of equation (1) has a basis of $n$ solution atoms which are determined by Euler’s theorem.
### Theorem 5 (How to Apply Euler’s Theorem)

<table>
<thead>
<tr>
<th>Factor dividing $p(r)$</th>
<th>Solution Atom(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r - 5$</td>
<td>$e^{5x}$</td>
</tr>
<tr>
<td>$(r + 7)^2$</td>
<td>$e^{-7x}$, $xe^{-7x}$</td>
</tr>
<tr>
<td>$(r + 7)^3$</td>
<td>$e^{-7x}$, $xe^{-7x}$, $x^2e^{-7x}$</td>
</tr>
<tr>
<td>$r$</td>
<td>$e^{0x}$</td>
</tr>
<tr>
<td>$r^2$</td>
<td>$e^{0x}$ and $xe^{0x}$</td>
</tr>
<tr>
<td>$r^3$</td>
<td>$1$, $x$ and $x^2$ [$e^{0x} = 1$]</td>
</tr>
<tr>
<td>$r - 5i$</td>
<td>$\cos 5x$ and $\sin 5x$</td>
</tr>
<tr>
<td>$(r + 3i)^2$</td>
<td>$\cos 3x$, $x \cos 3x$, $\sin 3x$, $x \sin 3x$</td>
</tr>
<tr>
<td>$(r - 2 + 3i)^2$</td>
<td>$e^{2x} \cos 3x$, $xe^{2x} \cos 3x$, $e^{2x} \sin 3x$, $xe^{2x} \sin 3x$</td>
</tr>
</tbody>
</table>
Example 1. Solve $y''' = 0$.  

**Solution:** $p(r) = r^3$ implies $1$ is a base atom and then $1, x, x^2$ are solution atoms. They are independent, hence form a basis for the 3-dimensional solution space. Then $y = c_1 + c_2x + c_3x^2$.

Example 2. Solve $y'' + 4y = 0$.  

**Solution:** $p(r) = r^2 + 4$ implies base atoms $\cos 2x$ and $\sin 2x$. They are a basis for the 2-dimensional solution space with $y = c_1 \cos 2x + c_2 \sin 2x$.

Example 3. Solve $y'' + 2y' = 0$.  

**Solution:** $p(r) = r^2 + 2r$ implies $1, e^{-2x}$ are base solution atoms. These independent atoms form a basis. Then $y = c_1 + c_2 e^{-2x}$.

Example 4. Solve $y^{(4)} + 4y'' = 0$.  

**Solution:** $p(r) = r^4 + 4r^2 = r^2(r^2 + 4)$ implies the four atoms $1, x, \cos 2x, \sin 2x$ are solutions. Then $y = c_1 + c_2x + c_3 \cos 2x + c_3 \sin 2x$.

Example 5. Solve the differential equation if $p(r) = (r^3 - r^2)(r^2 - 1)(r^2 + 4)^2$.  

**Solution:** The distinct factors of $p(r)$ are $r^2, (r - 1)^2, r + 1, (r - 2i)^2, (r + 2i)^2$. Euler’s theorem implies the DE has nine solution atoms $1, x, e^x, xe^x, e^{-x}, \cos 2x, x \cos 2x, \sin 2x, x \sin 2x$. Then $y$ is a linear combination of these atoms.