How to Solve Linear Differential Equations

• Definition: Base Atom, Atom

• Independence of Atoms

• Construction of the General Solution from a List of Distinct Atoms

• Euler’s Theorems
  – Euler’s Basic Theorem
  – Euler’s Multiplicity Theorem
  – A Shortcut Method

• Examples

• Main Theorems on Atoms and Linear Differential Equations
Solution Atoms of Homogeneous Linear Differential Equations

Definition

• A base atom is one of 1, $e^{ax}$, $\cos bx$, $\sin bx$, $e^{ax} \cos bx$, $e^{ax} \sin bx$, with $b > 0$ and $a \neq 0$.

• An atom equals $x^n$ times a base atom, where $n \geq 0$ is an integer.

Details and Remarks

• Euler’s formula $e^{i\theta} = \cos \theta + i \sin \theta$ implies that an atom is constructed from the complex expression $x^n e^{ax + ibx}$ by taking real and imaginary parts.

• The powers $1, x, x^2, \ldots, x^k$ are atoms.

• The term that makes up an atom has coefficient 1, therefore $2e^x$ is not an atom, but the 2 can be stripped off to create the atom $e^x$. Zero is not an atom. Linear combinations like $2x + 3x^2$ are not atoms, but the individual terms $x$ and $x^2$ are indeed atoms. Terms like $-e^x$, $e^{-x^2}$, $x^{5/2} \cos x$, $\ln |x|$ and $x/(1 + x^2)$ are not atoms.
Linear algebra defines a list of functions $f_1, \ldots, f_k$ to be **linearly independent** if and only if the representation of the zero function as a linear combination of the listed functions is uniquely represented, that is,

$$0 = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x)$$

for all $x$, implies $c_1 = c_2 = \cdots = c_k = 0$.

**Independence and Atoms**

**Theorem 1 (Atoms are Independent)**
A list of finitely many distinct atoms is linearly independent.

**Theorem 2 (Powers are Independent)**
The list of distinct atoms $1, x, x^2, \ldots, x^k$ is linearly independent. And all of its sublists are linearly independent.
Construction of the General Solution from a List of Distinct Atoms

- **Picard’s theorem** says that the homogeneous constant-coefficient linear differential equation

\[ y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1y' + p_0 = 0 \]

has solution space \( S \) of dimension \( n \). Picard’s theorem reduces the general solution problem to finding \( n \) linearly independent solutions.

- **Euler’s theorem** *infra* says that the required \( n \) independent solutions can be taken to be atoms. The theorem explains how to construct a list of distinct atoms, each of which is a solution of the differential equation, from the roots of the characteristic equation [Definition: The left side is the **characteristic polynomial**.]

\[ r^n + p_{n-1}r^{n-1} + \cdots + p_1r + p_0 = 0. \]

- The **Fundamental Theorem of Algebra** is part of the Doctoral thesis of Carl Friedrich Gauss (1777–1855): *An nth order polynomial equation has exactly n roots, real or complex, counted according to multiplicities.* Therefore, the characteristic equation has exactly \( n \) roots, counting multiplicities.

- **General Solution.** Because the list of atoms constructed by Euler’s theorem has \( n \) distinct elements, then this list of independent atoms forms a **basis** for the general solution of the differential equation, giving

\[ y = c_1(\text{atom 1}) + \cdots + c_n(\text{atom } n). \]

Symbols \( c_1, \ldots, c_n \) are arbitrary coefficients. In particular, each atom listed is itself a solution of the differential equation.
Euler’s Basic Theorem

Theorem 3 (L. Euler)
The exponential \( y = e^{r_1 x} \) is a solution of a constant-coefficient linear homogeneous differential of the \( n \)th order if and only if \( r = r_1 \) is a root of the characteristic equation.

- If \( r_1 = a \) is a real root, then Euler’s Theorem constructs one real solution atom \( e^{ax} \).
- If \( r_1 = a + ib \) is a complex root (\( b > 0 \)), then Euler’s Theorem constructs two real solution atoms
  \[ e^{ax} \cos bx, \quad e^{ax} \sin bx. \]

Derivation is from Euler’s complex solution

\[ e^{r_1 x} = e^{ax} \cos bx + ie^{ax} \sin bx. \]

The real and imaginary parts of this complex solution are real solutions of the differential equation. The conjugate pair of roots \( a + ib \) and \( a - ib \) produce the same two atoms, which explains the simplification above.
Euler’s Multiplicity Theorem

**Definition.** A root \( r = r_1 \) of a polynomial equation \( p(r) = 0 \) has **multiplicity** \( k \) provided \( (r - r_1)^k \) divides \( p(r) \) but \( (r - r_1)^{k+1} \) does not divide \( p(r) \). The calculus equivalent is \( \frac{d^j p}{dr^j}(r_1) = 0 \) for \( j = 0, \ldots, k - 1 \) and \( \frac{d^k p}{dr^k}(r_1) \neq 0 \).

**Theorem 4 (L. Euler)**

The expression \( y = x^k e^{r_1 x} \) is a solution of a constant-coefficient linear homogeneous differential of the \( n \)th order if and only if \( (r - r_1)^{k+1} \) divides the characteristic polynomial.

**A Shortcut for using Euler’s Theorems**

Given a real root \( r_1 \) or complex root \( r_1 = a + ib \), apply Euler’s first theorem to obtain the base atom \( e^{r_1 x} \), or the pair of base atoms \( e^{ax} \cos bx, e^{ax} \sin bx \). Multiply each base item by powers 1, \( x \), \( x^2 \), \ldots, until the number of atoms obtained equals the multiplicity of root \( r_1 \).
Atom List Examples

1. If root \( r = -3 \) has multiplicity 4, then the atom list is

\[ e^{-3x}, xe^{-3x}, x^2 e^{-3x}, x^3 e^{-3x}. \]

The list is constructed by multiplying the base atom \( e^{-3x} \) by powers 1, \( x \), \( x^2 \), \( x^3 \). The multiplicity 4 of the root equals the number of constructed atoms.

2. If \( r = -3 + 2i \) is a root of the characteristic equation, then the base atoms for this root (both \(-3 + 2i \) and \(-3 - 2i \) counted) are

\[ e^{-3x} \cos 2x, \quad e^{-3x} \sin 2x. \]

If root \( r = -3 + 2i \) has multiplicity 3, then the two real atoms are multiplied by 1, \( x \), \( x^2 \) to obtain a total of 6 atoms

\[ e^{-3x} \cos 2x, \quad xe^{-3x} \cos 2x, \quad x^2 e^{-3x} \cos 2x, \quad e^{-3x} \sin 2x, \quad xe^{-3x} \sin 2x, \quad x^2 e^{-3x} \sin 2x. \]

The number of atoms generated for each base atom is 3, which equals the multiplicity of the root \(-3 + 2i \).
Theorem 5 (Homogeneous Solution $y_h$ and Atoms)
Linear homogeneous $n$th order differential equations with constant coefficients have general solution $y_h(x)$ equal to a linear combination of $n$ distinct atoms.

Theorem 6 (Particular Solution $y_p$ and Atoms)
A linear non-homogeneous differential equation with constant coefficients $a$ having forcing term $f(x)$ equal to a linear combination of atoms has a particular solution $y_p(x)$ which is a linear combination of atoms.

Theorem 7 (General Solution $y$ and Atoms)
A linear non-homogeneous differential equation with constant coefficients having forcing term

$$ f(x) = \text{a linear combination of atoms} $$

has general solution

$$ y(x) = y_h(x) + y_p(x) = \text{a linear combination of atoms}. $$

Proofs
The first theorem follows from Picard’s theorem, Euler’s theorem and independence of atoms. The second follows from the method of undetermined coefficients, *infra*. The third theorem follows from the first two.