# How to Solve Linear Differential Equations

- Definition: Base Atom, Atom
- Independence of Atoms
- Construction of the General Solution from a List of Distinct Atoms
- Euler's Theorems
  - Euler's Basic Theorem
  - Euler's Multiplicity Theorem
  - A Shortcut Method
- Examples
- Main Theorems on Atoms and Linear Differential Equations

## Solution Atoms of Homogeneous Linear Differential Equations

### Definition

- A **base atom** is one of $1$, $e^{ax}$, $\cos bx$, $\sin bx$, $e^{ax}\cos bx$, $e^{ax}\sin bx$, with $b > 0$ and $a \neq 0$.
- An **atom** equals $x^n$ times a base atom, where $n \geq 0$ is an integer.

## Details and Remarks

- Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ implies that an atom is constructed from the complex expression $x^n e^{ax+ibx}$ by taking real and imaginary parts.
- The powers $1, x, x^2, \ldots, x^k$ are atoms.
- The term that makes up an atom has coefficient $1$, therefore $2e^x$ is not an atom, but the $2$ can be stripped off to create the atom $e^x$. Zero is not an atom. Linear combinations like $2x + 3x^2$ are not atoms, but the individual terms $x$ and $x^2$ are indeed atoms. Terms like $-e^x$, $e^{-x^2}$, $x^{5/2}\cos x$, $\ln|x|$ and $x/(1+x^2)$ are not atoms.

## Independence

Linear algebra defines a list of functions $f_1, \ldots, f_k$ to be **linearly independent** if and only if the representation of the zero function as a linear combination of the listed functions is uniquely represented, that is,

$$0 = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) \quad \text{for all } x$$

implies $c_1 = c_2 = \cdots = c_k = 0$.

## Independence and Atoms

**Theorem 1 (Atoms are Independent)**
A list of finitely many distinct atoms is linearly independent.

**Theorem 2 (Powers are Independent)**
The list of distinct atoms $1, x, x^2, \ldots, x^k$ is linearly independent. And all of its sublists are linearly independent.

## Construction of the General Solution from a List of Distinct Atoms

- **Picard's theorem** says that the homogeneous constant-coefficient linear differential equation

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0$$

  has solution space $S$ of dimension $n$. Picard's theorem reduces the general solution problem to finding $n$ linearly independent solutions.

- **Euler's theorem** *infra* says that the required $n$ independent solutions can be taken to be atoms. The theorem explains how to construct a list of distinct atoms, each of which is a solution of the differential equation, from the roots of the characteristic equation [Definition: The left side is the **characteristic polynomial**.]

$$r^n + p_{n-1}r^{n-1} + \cdots + p_1 r + p_0 = 0.$$

- The **Fundamental Theorem of Algebra** is part of the Doctoral thesis of Carl Friedrich Gauss (1777–1855): *An $n$th order polynomial equation has exactly $n$ roots, real or complex, counted according to multiplicities*. Therefore, the characteristic equation has exactly $n$ roots, counting multiplicities.

- **General Solution**. Because the list of atoms constructed by Euler's theorem has $n$ distinct elements, then this list of independent atoms forms a **basis** for the general solution of the differential equation, giving

$$y = c_1(\text{atom } 1) + \cdots + c_n(\text{atom } n).$$

  Symbols $c_1, \ldots, c_n$ are arbitrary coefficients. In particular, each atom listed is itself a solution of the differential equation.

**Theorem 3 (L. Euler)**
The exponential $y = e^{r_1 x}$ is a solution of a constant-coefficient linear homogeneous differential of the $n$th order if and only if $r = r_1$ is a root of the characteristic equation.

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- If $r_1 = a$ is a real root, then Euler's Theorem constructs *one real solution atom* $e^{ax}$.
- If $r_1 = a + ib$ is a complex root ($b > 0$), then Euler's Theorem constructs *two real solution atoms*
$$e^{ax}\cos bx, \quad e^{ax}\sin bx.$$

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Derivation is from Euler's complex solution

$$e^{r_1 x} = e^{ax}\cos bx + ie^{ax}\sin bx.$$

The real and imaginary parts of this complex solution are real solutions of the differential equation. The conjugate pair of roots $a + ib$ and $a - ib$ produce the same two atoms, which explains the simplification above.

## Euler's Multiplicity Theorem

**Definition**. A root $r = r_1$ of a polynomial equation $p(r) = 0$ has **multiplicity** $k$ provided $(r - r_1)^k$ divides $p(r)$ but $(r - r_1)^{k+1}$ does not divide $p(r)$. The calculus equivalent is $\frac{d^j p}{dr^j}(r_1) = 0$ for $j = 0, .., k - 1$ and $\frac{d^k p}{dr^k}(r_1) \neq 0$.

**Theorem 4 (L. Euler)**
The expression $y = x^k e^{r_1 x}$ is a solution of a constant-coefficient linear homogeneous differential of the $n$th order if and only if $(r - r_1)^{k+1}$ divides the characteristic polynomial.

## A Shortcut for using Euler's Theorems

Given a real root $r_1$ or complex root $r_1 = a + ib$, apply Euler's first theorem to obtain the base atom $e^{r_1 x}$, or the pair of base atoms $e^{ax}\cos bx, e^{ax}\sin bx$. Multiply each base item by powers $1$, $x$, $x^2$, ..., until the number of atoms obtained equals the multiplicity of root $r_1$.

## Atom List Examples

1. If root $r = -3$ has multiplicity $4$, then the atom list is

$$e^{-3x}, xe^{-3x}, x^2e^{-3x}, x^3e^{-3x}.$$

   The list is constructed by multiplying the base atom $e^{-3x}$ by powers $1$, $x$, $x^2$, $x^3$. The multiplicity $4$ of the root equals the number of constructed atoms.

2. If $r = -3 + 2i$ is a root of the characteristic equation, then the base atoms for this root (both $-3 + 2i$ and $-3 - 2i$ counted) are

$$e^{-3x}\cos 2x, \quad e^{-3x}\sin 2x.$$

   If root $r = -3 + 2i$ has multiplicity $3$, then the two real atoms are multiplied by $1$, $x$, $x^2$ to obtain a total of $6$ atoms

$$\begin{gathered} e^{-3x}\cos 2x,\;\; xe^{-3x}\cos 2x,\;\; x^2e^{-3x}\cos 2x, \\ e^{-3x}\sin 2x,\;\; xe^{-3x}\sin 2x,\;\; x^2e^{-3x}\sin 2x. \end{gathered}$$

   The number of atoms generated for each base atom is $3$, which equals the multiplicity of the root $-3 + 2i$.

**Theorem 5 (Homogeneous Solution $y_h$ and Atoms)**
Linear homogeneous $n$th order differential equations with constant coefficients have general solution $y_h(x)$ equal to a linear combination of $n$ distinct atoms.

**Theorem 6 (Particular Solution $y_p$ and Atoms)**
A linear non-homogeneous differential equation with constant coefficients a having forcing term $f(x)$ equal to a linear combination of atoms has a particular solution $y_p(x)$ which is a linear combination of atoms.

**Theorem 7 (General Solution $y$ and Atoms)**
A linear non-homogeneous differential equation with constant coefficients having forcing term

$$f(x) = \text{ a linear combination of atoms}$$

has general solution

$$y(x) = y_h(x) + y_p(x) = \text{ a linear combination of atoms.}$$

## Proofs

The first theorem follows from Picard's theorem, Euler's theorem and independence of atoms. The second follows from the method of undetermined coefficients, *infra*. The third theorem follows from the first two.