Differential Equations and Linear Algebra

2250-1 at 7:30am on 27 April 2012

Instructions. The time allowed is 120 minutes. The examination consists of eight problems, one for each of chapters 3, 4, 5, 6, 7, 8, 9, 10, each problem with multiple parts. A chapter represents 15 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c), $\cdots$. Each chapter (3 to 10) adds at most 100 towards the maximum final exam score of 800. The final exam grade is reported as a percentage 0 to 100, as follows:

$$\text{Final Exam Grade} = \frac{\text{Sum of scores on eight chapters}}{8}.$$

- Calculators, books, notes, computers and electronics are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader. More credit is possible if you write something.
- Answer checks are not expected and they are not required. First drafts are expected, not complete presentations.
- Please prepare **exactly one** stapled package of all eight chapters, organized by chapter. All scratch work for a chapter must appear in order. Any work stapled out of order could be missed, due to multiple graders.
- The graded exams will be in a box outside 113 JWB; you will pick up one stapled package.
- Records will be posted at the Registrar's web site on **WEBCT**. Recording errors can be reported by email, hopefully as soon as discovered, but also even weeks after grades are posted.

Final Grade. The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90 + 91 + 92 + 89 + 89}{5} = 90.2.$$

Dailies, both maple and math together, count 30% of the final grade. The course average is computed from the formula

$$\text{Course Average} = \frac{70}{100}(\text{Exam Average}) + \frac{30}{100}(\text{Dailies Average}).$$

Please recycle this page or keep it for your records.

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Scores
Ch3.
Ch4.
Ch5.
Ch6.
Ch7.
Ch8.
Ch9.
Ch10.

Ch3. (Linear Systems and Matrices) Complete all problems.

[20%] **Ch3(a)**: Incorrect answers lose all credit.
Ch3(a) Part 1. [10%]: True or False (circle the correct answer):
If the 3×3 matrix A is given, then invertible matrices $E_1, E_2, \ldots, E_k$ exist such that $E_k \cdots E_2 E_1 A = U$ where U is upper triangular.
Ch3(a) Part 2. [10%]: True or False (circle the correct answer):
If a 3×3 matrix A has no inverse, and $\vec{b}$ is a given vector, then the equation $A\vec{x} = \vec{b}$ has no solution $\vec{x}$.

Answer: True. False.

[40%] **Ch3(b)**: Determine which values of k correspond to infinitely many solutions for the system $A\vec{x} = \vec{b}$ given by

$$A = \begin{pmatrix} 1 & 4 & k \\ 0 & k-2 & k-3 \\ 1 & 4 & 3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ -1 \\ k \end{pmatrix}.$$

Answer: (1) There is a unique solution for $\det(A) \neq 0$, which implies $k \neq 2$ and $k \neq 3$. Therefore, the answer for infinitely many solutions, if there is one, must have either $k = 2$ or $k = 3$ or both. Elimination methods applied to the augmented matrix $C =< A|\vec{b} >$, using only one combo and no swap or multiply, give the frame $\begin{pmatrix} 1 & 4 & k & 1 \\ 0 & k-2 & 0 & k-2 \\ 0 & 0 & 3-k & k-1 \end{pmatrix}$. The no solution case must have a signal equation. The infinite solution case must have a free variable but no signal equation. Then (2) No solution for $k = 3$ [equation 3 is a signal equation]; (3) Infinitely many solutions for $k = 2$ [a free variable x_2 but no signal equation].

[40%] **Ch3(c)**: Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Find the value of x_3 by Cramer's Rule in the system $A\vec{x} = \vec{b}$.

Answer: $x_3 = \Delta_3/\Delta$, $\Delta_3 = \det \begin{pmatrix} -2 & 3 & 1 \\ 0 & -2 & 2 \\ 1 & 0 & 3 \end{pmatrix} = 20$, $\Delta = \det(A) = 12$, $x_3 = \frac{20}{12} = \frac{5}{3}$.

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Ch4. (Vector Spaces) Complete all problems.

[20%] **Ch4(a)**: Check the independence tests which apply to prove that vectors $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are independent in the vector space V of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ (the vector space $V = \mathcal{R}^3$).

☐	**Wronskian test**	Wronskian of functions f_1, f_2, f_3 nonzero at $x = x_0$ implies independence of f_1, f_2, f_3.
☐	**Rank test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Atom test**	Any finite set of distinct atoms is independent.
☐	**Pivot test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.

Answer: The first and fourth fail to apply, because these tests are about functions, not fixed vectors.

[20%] **Ch4(b)**: Consider the homogenous system $A\vec{x} = \vec{0}$. The Rank-Nullity Theorem says that the rank plus the nullity equals the number of variables. The number of variables equals the number of columns of A. Give an example of a matrix A with four columns that has nullity 2.

Answer: Let $\vec{v}_1$, $\vec{v}_2$ be any two columns of the identity and let $\vec{v}_3 = \vec{v}_4$ each be the zero vector. Define A to be the augmented matrix of these four vectors. Then A has 4 columns, rank 2 and nullity 2.

[30%] **Ch4(c)**: Let V be the vector space of all twice continuously differentiable functions. Let S be the set of all solutions y of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$. Find two independent solutions y_1, y_2 such that $S = \mathbf{span}(y_1, y_2)$. This calculation proves that S is a subspace of V by the Span Theorem.

Answer: The roots of the characteristic equation are $r = 0, 1$. Then atoms e^{0x}, e^x make the general solution $y = c_1 + c_2 e^x$. This means that $S = \mathbf{span}(1, e^x)$, because the span is the set of all linear combinations of 1, e^x, which equals the set of all solutions y obtained from the general solution expression $c_1 + c_2 e^x$.

[30%] **Ch4(d)**: The 4×7 matrix A below has some independent columns. Report the independent columns of A, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & -2 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 6 & 0 & 0 & 6 & 0 & 0 & 3 \end{pmatrix}$$

Answer: Find $\mathbf{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$. The pivot columns are 1 and 4.

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Ch5. (Linear Equations of Higher Order) Complete all problems.

[20%] **Ch5(a)**: Find the characteristic equation of a higher order linear homogeneous differential equation with constant coefficients, of minimum order, such that $y = x + x^3 + 10e^{-2x} + 3x\sin x$ is a solution.

Answer: The atoms x, x^2, e^{-x}, $x\sin x$ correspond to roots $0, 0, 0, 0, -2, i, -i, i, -i$. The factor theorem implies the characteristic polynomial should be $r^4(r+2)(r^2+1)^2$.

[20%] **Ch5(b)**: Determine a *basis of solutions* of a homogeneous constant-coefficient linear differential equation, given it has characteristic equation

$$r(r^2 + 5r)^2((r+1)^2 + 3)^2 = 0.$$

Answer: The roots are $0, 0, 0, -5, -5, -1\pm\sqrt{3}i, -1\pm\sqrt{3}i$. By Euler's theorem, a basis is the set of atoms for these roots: $1, x, x^2, e^{-5x}, xe^{-5x}, e^{-x}\cos(\sqrt{3}x), e^{-x}\sin(\sqrt{3}x), xe^{-x}\cos(\sqrt{3}x), xe^{-x}\sin(\sqrt{3}x)$.

[30%] **Ch5(c)**: Find the steady-state periodic solution for the equation

$$x'' + 4x' + 13x = 40\cos(t).$$

It is known that this solution equals the undetermined coefficients solution for a particular solution $x_p(t)$.

Answer: Use undetermined coefficients trial solution $x = d_1\cos t + d_2\sin t$. Then $d_1 = 3$, $d_2 = 1$.

[30%] **Ch5(d)**: Determine the **shortest** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$\frac{d^4y}{dx^4} + 9\frac{d^2y}{dx^2} = 2xe^{3x} + 5x\sin 3x$$

Answer: The homogeneous problem has roots $0, 0, \pm 3i$ with atoms $1, x, \cos 3x, \sin 3x$. The trial solution is constructed initially from $f(x) = 2xe^{3x} + 3x\cos 3x$, which has seven atoms in a list in three groups (1) e^{3x}, xe^{3x}; (2) $\cos 3x, x\cos 3x$; (3) $\sin 3x, x\sin 3x$. Conflicts with the homogeneous equation atoms causes a repair of groups (2), (3) making the new groups (1) e^{3x}, xe^{3x}; (2) $x\cos 3x, x^2\cos 3x$; (3) $x\sin 3x, x^2\sin 3x$. Then the shortest trial solution is a linear combination of the six atoms in the corrected list.

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Ch6. (Eigenvalues and Eigenvectors) Complete all problems.

[30%] **Ch6(a)**: Let $A = \begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}$. Circle possible eigenpairs of A.

$$\left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right), \quad \left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(-1, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right).$$

Answer: The second and the third, because the test $A\vec{x} = \lambda\vec{x}$ passes in both cases.

[40%] **Ch6(b)**: Find the eigenvalues of the matrix B:

$$B = \begin{pmatrix} 2 & 4 & -1 & 0 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

Answer: The characteristic polynomial is $\det(B - rI) = (2-r)(5-r)(5-r)(3-r)$. The eigenvalues are $2, 3, 5, 5$.

It is possible to directly find the eigenvalues of B by cofactor expansion of $|B - rI|$. This is the expected method. Expand $|B - rI|$ by cofactors on column 1. The calculation reduces to a 3×3 determinant. Expand the 3×3 along column 1 to reduce the calculation to a 2×2, which evaluates as a quadratic by Sarrus' Rule.

An alternate method is described below, which depends upon a determinant product theorem for special block matrices, such as encountered in this example.

Matrix B is a block matrix $B = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline 0 & B_3 \end{array}\right)$, where B_1, B_2, B_3 are all 2×2 matrices. Then $B - rI = \left(\begin{array}{c|c} B_1 - rI & B_2 \\ \hline 0 & B_3 - rI \end{array}\right)$. Using the determinant product theorem for such special block matrices (zero in the left lower block) gives $|B - rI| = |B_1 - rI||B_3 - rI|$. So the answer is that B has eigenvalues equal to the eigenvalues of B_1 and B_3. These are quickly found by Sarrus' Rule applied to the two 2×2 determinants $|B_1 - rI| = (2-r)(5-r)$ and $|B_3 - rI| = r^2 - 8r + 15 = (5-r)(3-r)$.

[30%] **Ch6(c)**: The matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix}$ has eigenvalues $2, 2, 2$ and it is **not diagonalizable**.
Find all eigenvectors for $\lambda = 2$.

Answer: Because $A - 2I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ has last frame $B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then there are two eigenpairs for $\lambda = 2$, with eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

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Ch7. (Linear Systems of Differential Equations) Complete all problems.

[30%] **Ch7(a)**: Let A be an $n \times n$ matrix of real numbers. When A is diagonalizable, then eigenanalysis applies to solve the system $\vec{u}' = A\vec{u}$. State two methods which can be used to solve this system when the matrix is not diagonalizable.

Answer: Instead of eigenanalysis we can use Cayley-Hamilton-Ziebur, Laplace resolvent, exponential matrix.

[30%] **Ch7(b)**: Solve the 3×3 differential system $\vec{u}' = A\vec{u}$, given matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Use the most efficient method that you know.

Answer: The matrix is triangular, so the theory of linear cascades applies. First, $x_3' = 0$, then $x_3 = c_1$. Back-substitute: $x_3' = 0$ implies $x_3 = c_1$ and $x_2' = 2x_3 = 2c_1$ implies $x_2 = 2c_1 t + c_2$. Then $x_1' = x_2 = c_1 t + c_2$, which implies $x_1 = c_1 t^2/2 + c_2 t + c_3$.

Does Cayley-Hamilton-Ziebur work? Yes, it always works. The roots of $|A - rI| = 0$ are $r = 0, 0, 0$, which implies atoms $1, t, t^2$ and then $\vec{u} = \vec{d}_1 + \vec{d}_2 t + \vec{d}_3 t^2$. Differentiate this relation two times and set $t = 0$ in the resulting 3 equations in 3 vector unknowns $\vec{d}_1, \vec{d}_2, \vec{d}_3$. Solve the system for the unknown vectors, in terms of A and $\vec{u}(0) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \equiv \vec{c}$. You will get $\vec{c} = \vec{d}_1, A\vec{c} = \vec{d}_2, A^2\vec{c} = \vec{d}_3$. Then $\vec{u}(t) = \left(I + At + A^2t^2\right)\vec{c}$. This answer is slightly different than what was obtained by linear cascades, and it takes more time to produce.

Does the Eigenanalysis method apply? The roots of the characteristic equation are $0, 0, 0$, and the corresponding atoms are $1, x, x^2$. But $A - 0I$ is just A, which has rank 2 and nullity 1. There is only one eigenpair: A is not diagonalizable. The eigenanalysis method *does not apply*.

Laplace theory? Yes, it always works. Begin with the Laplace resolvent equation $(sI - A)\mathcal{L}(\vec{u}) = \vec{u}(0) = \vec{c}$. Solve as $\mathcal{L}(\vec{u}) = (sI - A)^{-1}\vec{c}$. Using the backward Laplace table on each entry in $(sI - A)^{-1}$ gives the answer $\vec{u}(t) = \begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$. None of this is easy, and the preferred method is linear cascades.

[40%] **Ch7(c)**: Apply the eigenanalysis method to solve the system $\vec{x}' = A\vec{x}$, given

$$A = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 0 & 0 & -4 \end{pmatrix}$$

Answer: The eigenpairs of A are $(-5, \vec{v}_1), (-3, \vec{v}_2), (-4, \vec{v}_3)$ where

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Then

$$\vec{u}(t) = c_1 e^{-5t}\vec{v}_1 + c_1 e^{-3t}\vec{v}_2 + c_1 e^{-4t}\vec{v}_3.$$

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Ch8. (Matrix Exponential) Complete all problems.

[40%] **Ch8(a)**: Using any method in the lectures or the textbook, display the matrix exponential e^{Bt}, for

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}.$$

Answer: The exponential matrix is $\begin{pmatrix} 1 & \frac{1}{2}e^{2t} - \frac{1}{2} \\ 0 & e^{2t} \end{pmatrix}$.

Method 1. We find a fundamental matrix $Z(t)$. It is done by solving the matrix system $\vec{u}' = B\vec{u}$ with Ziebur's shortcut. Then $x = c_1 + c_2e^{2t}$, $y = x' = 2c_2e^{2t}$. Write $\vec{u} = Z(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ where $Z(t) = \begin{pmatrix} 1 & e^{2t} \\ 0 & 2e^{2t} \end{pmatrix}$. Then

$$e^{Bt} = Z(t)Z(0)^{-1} = \begin{pmatrix} 1 & e^{2t} \\ 0 & 2e^{2t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{-1}.$$

Method 2. Putzer's formula can also be used, $e^{Bt} = e^{\lambda_1 t}I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(B - \lambda_1 I)$. The eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2$ are found from $\det(B - \lambda I) = \lambda^2 - 2\lambda = 0$. Then Putzer's formula becomes $e^{Bt} = I + \frac{1-e^{2t}}{0-2}(B - 0I)$. Expanding gives the reported answer.

Method 3. A third method would use the formula $e^{Bt} = \mathcal{L}^{-1}((sI - B)^{-1})$ to find the answer in a sequence of Laplace steps.

[30%] **Ch8(b)**: Consider the 2×2 system

$$\begin{aligned} x' &= 2x, \\ y' &= -2y, \\ x(0) &= 1, \quad y(0) = 2. \end{aligned}$$

Solve the system as a matrix problem $\vec{u}' = A\vec{u}$ for $\vec{u}$, using the matrix exponential e^{At}.

Answer: Let $A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$. The answer for e^{At} can be obtained quickly from the theorem $e^{\mathbf{diag}(a,b)t} = \mathbf{diag}(e^{at}, e^{bt})$, giving the answer $e^{At} = \mathbf{diag}(e^{2t}, e^{-2t})$. Then use $\vec{u}(t) = e^{At}\vec{u}(0)$, which implies $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ 2e^{-2t} \end{pmatrix}$.

[30%] **Ch8(c)**: Display the matrix form of variation of parameters for the 2×2 system. Then integrate to find one particular solution.

$$\begin{aligned} x' &= 2x, \\ y' &= -2y + 1. \end{aligned}$$

Answer: Variation of parameters is $\vec{u}_p(t) = e^{At} \int_0^t e^{-As} \begin{pmatrix} 0 \\ 1 \end{pmatrix} ds$. Then $e^{At} = \mathbf{diag}(e^{2t}, e^{-2t})$ from the previous problem. Substitute $t \to -s$ to obtain $e^{-As} = \mathbf{diag}(e^{-2s}, e^{2s})$. The integration step is

$$\int_0^t \mathbf{diag}(e^{-2s}, e^{2s}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} ds = \int_0^t \begin{pmatrix} 0 \\ e^{2s} \end{pmatrix} ds = \begin{pmatrix} 0 \\ \frac{1}{2}e^{2t} - \frac{1}{2} \end{pmatrix}.$$

Finally, $\vec{u}_p(t) = \mathbf{diag}(e^{2t}, e^{-2t}) \begin{pmatrix} 0 \\ \frac{1}{2}e^{2t} - \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} - \frac{1}{2}e^{-2t} \end{pmatrix}$.

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Ch9. (Nonlinear Systems) Complete all problems.

[30%] **Ch9(a)**:
Determine whether the equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node.

$$\vec{u}' = \begin{pmatrix} 5 & 10 \\ -4 & -7 \end{pmatrix} \vec{u}$$

Answer: It is a stable spiral. Details: The eigenvalues of A are roots of $r^2+2r+5 = (r+1)^2+4 = 0$, which are complex conjugate roots $-1 \pm 2i$. Rotation eliminates the saddle and node. Finally, the atoms $e^{-t}\cos 2t$, $e^{-t}\sin 2t$ have limit zero at $t = \infty$, therefore the system is stable at $t = \infty$. So it must be a spiral [centers have no exponentials]. Report: **stable spiral**.

[30%] **Ch9(b)**: Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - y^2 - y + 16, \\ y' &= 2x^2 - 2xy. \end{aligned}$$

An equilibrium point is $x = 4$, $y = 4$. Compute the Jacobian matrix $A = J(4,4)$ of the linearized system at this equilibrium point.

Answer: The Jacobian is $J(x,y) = \begin{pmatrix} 1 & -2y-1 \\ 4x-2y & -2x \end{pmatrix}$. Then $A = J(4,4) = \begin{pmatrix} 1 & -9 \\ 8 & -8 \end{pmatrix}$.

[40%] **Ch9(c)**: Consider the nonlinear dynamical system

$$\begin{aligned} x' &= 4x - 4y + 9 - x^2, \\ y' &= 3x - 3y. \end{aligned}$$

At equilibrium point $x = 3$, $y = 3$, the Jacobian matrix is $A = J(3,3) = \begin{pmatrix} -2 & -4 \\ 3 & -3 \end{pmatrix}$.

(1) Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the linear system $\vec{u}' = A\vec{u}$.

(2) Apply a theorem to classify $x = 3$, $y = 3$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem.

Answer: (1) The Jacobian is $J(x,y) = \begin{pmatrix} 4-2x & -4 \\ 3 & -3 \end{pmatrix}$. Then $A = J(3,3) = \begin{pmatrix} -2 & -4 \\ 3 & -3 \end{pmatrix}$.
The eigenvalues of A are found from $r^2+5r+18 = 0$, giving complex conjugate roots $-\frac{5}{2} \pm (\frac{1}{2})\sqrt{47}$. Because both atoms have trig functions, rotation happens, and the classification must be a center or spiral. The atoms limit zero at $t = \infty$, therefore it is a spiral and we report an **stable spiral** for the linear problem $\vec{u}' = A\vec{u}$ at equilibrium $\vec{u} = \vec{0}$. (2) Theorem 2 in 9.2 applies to say that the same is true for the nonlinear system: **stable spiral** at $x = 3$, $y = 3$.

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Ch10. (Laplace Transform Methods) Complete all problems.

It is assumed that you know the minimum forward Laplace integral table and the 8 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

[40%] **Ch10(a)**: Fill in the blank spaces in the Laplace tables. Each wrong answer subtracts 3 points from the total of 40.

$f(t)$					
$\mathcal{L}(f(t))$	$\dfrac{1}{s^4}$	$\dfrac{1}{s+4}$	$\dfrac{2}{s^2+9}$	$\dfrac{2s}{s^2+1}$	$\dfrac{e^{-s}}{s}$

$f(t)$	t^5	te^{-t}	$e^t\cos 2t$	t^2e^{2t}	$t\sin t$
$\mathcal{L}(f(t))$					

Answer: First table left to right: $\dfrac{t^3}{6}$, e^{-4t}, $\frac{2}{3}\sin 3t$, $2\cos t$, $\mathbf{step}(t-1)$. Function $\mathbf{step}(t)$ is the unit step $u(t)$ of the textbook, $\mathbf{step}(t)=1$ for $t\ge 0$, zero elsewhere.
Second table left to right: $\dfrac{5!}{s^6}$, $\dfrac{1}{(s+1)^2}$, $\left.\dfrac{s}{s^2+4}\right|_{s\to(s-1)} = \dfrac{s-1}{(s-1)^2+4}$, $\left.\dfrac{2}{s^3}\right|_{s\to(s-2)} = \dfrac{2}{(s-2)^3}$, $-\dfrac{d}{ds}\dfrac{1}{s^2+1} = \dfrac{2s}{(s^2+1)^2}$.

[30%] **Ch10(b)**: Compute $\mathcal{L}(f(t))$ for $f(t)=\sin(t)$ on $\pi < t < \infty$ and $f(t)=0$ elsewhere.

Answer: Define $\mathbf{step}(t)$ to be the unit step. Use $f(t)=\sin(t)\,\mathbf{step}(t-\pi)$ and the second shifting theorem. Then $\mathcal{L}(\sin(t)\,\mathbf{step}(t-\pi)) = e^{-\pi s}\mathcal{L}(\sin(t)|_{t\to t+\pi}) = e^{-\pi s}\mathcal{L}(\sin(t+\pi)) = e^{-\pi s}\mathcal{L}(\sin(t)\cos(\pi)+\sin(\pi)\cos(t)) = e^{-\pi s}\mathcal{L}(-\sin(t)) = e^{-\pi s}\frac{-1}{s^2+1}$. The answer: $\mathcal{L}(f(t)) = \frac{-e^{-pis}}{s^2+1}$.

[30%] **Ch10(c)**: Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{e^{-2s}}{s(s-2)}$.

Answer: First write $\frac{2}{s(s-2)} = \frac{-1}{s} + \frac{1}{s-2} = \mathcal{L}(-1+e^{2t})$ using partial fractions. We use $\mathcal{L}(\frac{1}{2}-\frac{1}{2}e^{2t}) = \frac{1}{s(s-2)}$ in a moment. Use the second shifting theorem, $e^{-as}\mathcal{L}(h(t)) = \mathcal{L}(h(t-a)\,\mathbf{step}(t-a))$. Then $\mathcal{L}(f(t)) = e^{-2s}\mathcal{L}(-\frac{1}{2}+\frac{1}{2}e^{2t}) = \mathcal{L}(-1+e^{2u}|_{u=t-2}\,\mathbf{step}(t-2)) = \mathcal{L}((-1+e^{2t-4})\,\mathbf{step}(t-2))$. Lerch's theorem implies $f(t) = \frac{1}{2}(-1+e^{2t-4})\,\mathbf{step}(t-2)$.

Place this page on top of all Ch10 work.