Name ________________________________________________

# Differential Equations and Linear Algebra 2250

**Midterm Exam 3**

**Version 1a, 19apr2012**

| Scores |
|---|
| 3. |
| 4. |
| 5. |

**Instructions**: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

**3.** **(Chapter 5)** Complete all.

**(3a)** [60%] The differential equation $\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 12x^2$ has homogeneous solution $y_h$ a linear combination of $1, x, e^x, e^{-x}$. Find a particular solution $y_p(x)$ of the form $y = d_1x^2 + d_2x^3 + d_3x^4$ by the method of undetermined coefficients (yes, find $d_1, d_2, d_3$).

Answer:

Solution $y_h$ is a linear combination of the atoms $1, x, e^x, e^{-x}$. A particular solution is $y_p = -x^4 - 12x^2$.

The atoms for $y^(4) - y'' = 0$ are found from $r^4 - r^2 = 0$ with roots $r = 0, 0, 1, -1$. The atoms in $f(x) = 12x^2$ are $1, x, x^2$. Because $1, x$ are solutions of the homogeneous equation, then the list $1, x, x^2$ from $f(x)$ is multiplied by $x^2$ to obtain the corrected list $x^2, x^3, x^4$. Then $y_p = d_1x^2 + d_2x^3 + d_3x^4$.

Substitute $y_p$ into the equation $y^(4) - y'' = 12x^2$ to get $24d_3 - (2d_1 + 6d_2x + 12d_3x^2) = 12x^2$. Matching coefficients of atoms gives $24d_3 - 2d_1 = 0, -6d_2 = 0, -12d_3 = 12$. Then $d_3 = -1, d_2 = 0, d_1 = -12$. Finally, $y_p = (-12)x^2 + (0)x^3 + (-1)x^4$.

**(3b)** [20%] Given $7x''(t) + 29x'(t) + 4x(t) = 0$, which represents a damped spring-mass system with $m = 7$, $c = 29$, $k = 4$, determine if the equation is over-damped , critically damped or under-damped. **To save time, do not solve for** $x(t)$!

Answer:

Use the quadratic formula to decide. The number under the radical sign in the formula, called the discriminant, is $b^2 - 4ac = 29^2 - 4(7)(4) = 729$, therefore there are distinct roots and the equation is **over-damped**. Alternatively, factor $7r^2 + 29r + 4$ to obtain roots $-1/7$, $-4$ and then classify as **over-damped**.

**(3c)** [20%] Consider the variation of parameters formula (33) in Edwards-Penney,

$$y_p(x) = y_1(x)\left(\int \frac{-y_2(x)f(x)}{W(x)}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(x)}dx\right).$$

Given the second order equation

$$2y''(x) + 4y'(x) + 3y(x) = 17\sin(x^2),$$

write the equations for the variables $y_1$, $y_2$, $f$.

**To save time, do not compute $W$ and do not write out $y_p$. Do not try to evaluate any integrals!**

Answer:

Variables are $y_1(x) = e^{-x}\cos(x/\sqrt{2})$, $y_2(x) = e^{-x}\sin(x/\sqrt{2})$, $f(x) = 17\sin(x^2)$.

Use this page to start your solution. Attach extra pages as needed.

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**4.** **(Chapter 5)** Complete all.

**(4a)** [60%] A homogeneous linear differential equation with constant coefficients has characteristic equation of order 8 with roots 0, $-1, -1, -1$, $3i, -3i$, $3i$, $-3i$, listed according to multiplicity. The corresponding non-homogeneous equation for unknown $y(x)$ has right side $f(x) = 0.01e^{-x} + 40x^2 + x\cos 3x + \sin 3x$. Determine the undetermined coefficients **shortest** trial solution for $y_p$.
**To save time, do not evaluate the undetermined coefficients and do not find $y_p(x)$! Undocumented detail or guessing earns no credit**.

Answer:

The atoms for roots of the characteristic equation are $1, e^{-x}, xe^{-x}, x^2e^{-x}, \cos 3x, x\cos 3x, \sin 3x, x\sin 3x$. The atom list for $f(x)$ is $e^{-x}$, $1, x, x^2$, $\cos 3x$, $x\cos 3x$, $\sin 3x$, $x\sin 3x$. This list of 8 atoms is broken into 4 groups, each group having exactly one base atom: (1) $1, x, x^2$, (2) $e^{-x}$, (3) $\cos 3x$, $x\cos 3x$, (4) $\sin 3x$, $x\sin 3x$. Each group contains a solution of the homogeneous equation. The modification rule is applied to groups 1 through 4. The trial solution is a linear combination of the replacement 8 atoms in the new list (1) $x, x^2, x^3$, (2) $x^3e^{-x}$, (3) $x^2\cos 3x$, $x^3\cos 3x$ (4) $x^2\sin 3x$, $x^3\sin 3x$.

**(4b)** [40%] Let $f(x) = (x + e^x)\sin x$. Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation which has $f(x)$ as a solution.

Answer:

Expand $f(x) = x\sin x + e^x \sin x$. Because $x\sin x$ is an atom for the differential equation if and only if $\sin x$, $x\sin x$, $\cos x$ $x\cos x$ are atoms, then the characteristic equation must have roots $i, -i, i, -i$, listing according to multiplicity (double complex root). Similarly, $e^x\sin x$ is an atom for the differential equation if and only if $1+i, 1-i$ are roots of the characteristic equation. Total of 6 roots with product of the factors $(r^2+1)^2((r-1)^2+1)$ equal to the 6th order characteristic polynomial.

Use this page to start your solution. Attach extra pages as needed.

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**5.** **(Chapter 6)** Complete all parts.

**(5a)** True and False. No details required.
[10%] True or False (circle the answer)
The matrix $A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ has no real eigenvalues.
[10%] True or False (circle the answer)
The matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ has only one eigenpair.
[10%] True or False (circle the answer)
The matrix $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ has a complex eigenvector $\vec{v} = \begin{pmatrix} i \\ -1 \end{pmatrix}$.

Answer:

False, False, False

**(5b)** [40%] Given $A = \begin{pmatrix} 2 & -1 & 2 & 0 \\ 0 & 5 & 0 & 0 \\ 3 & 1 & 3 & 0 \\ 2 & 3 & 4 & 0 \end{pmatrix}$, which has eigenvalues $5, 5, 0, 0$, display all solution details for finding the eigenvectors for eigenvalue 5.
**To save time, do not find the eigenvectors for eigenvalue** $0$.

Answer:

One frame sequence is required for $\lambda = 5$. Subtract 5 from the diagonal of $A$ to obtain a homogeneous system of the form $B\vec{x} = \vec{0}$. The sequence starts with $\begin{pmatrix} -3 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 1 & -2 & 0 \\ 2 & 3 & 4 & -5 \end{pmatrix}$, the last frame having two rows of zeros: $\begin{pmatrix} 1 & 0 & -\frac{10}{7} & \frac{5}{7} \\ 0 & 1 & \frac{16}{7} & -\frac{15}{7} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. There are two invented symbols $t_1$, $t_2$ in the last frame algorithm answer $x_1 = \frac{10}{7}t_1 - \frac{5}{7}t_2$, $x_2 = -\frac{16}{7}t_1 + \frac{15}{7}t_2$, $x_3 = t_1$, $x_4 = t_2$. Taking $\partial_{t_1}$ and $\partial_{t_2}$ gives two eigenvectors, $\begin{pmatrix} 10/7 \\ -16/7 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5/7 \\ 15/7 \\ 0 \\ 1 \end{pmatrix}$.

**(5c)** [30%] Find the matrices $P$, $D$ in the diagonalization equation $AP = PD$ for the matrix $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

Answer:

The eigenpairs are $\left(1, \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}\right)$, $\left(4, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$. Then $P = \begin{pmatrix} -1/2 & 1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

Use this page to start your solution. Attach extra pages as needed.