

Name KEY ver 1

Differential Equations and Linear Algebra 2250

Midterm Exam 2
Version 1, 22 Mar 2012

Scores

1
2
3

Instructions This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

F (The 3 Possibilities with Symbols)

Let a , b and c denote constants and consider the system of equations

$$\begin{pmatrix} c-a & -1 & 0 \\ a-c & 1 & a \\ 4a-2c & 3 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ b^2-b \\ b^2 \end{pmatrix}$$

- (a) [10%] Determine a , b and c such that the system has a unique solution.
- (b) [30%] Explain why $a = 0$ and $b \neq 0$ implies no solution. Ignore any other possible no solution cases.
- (c) [30%] Explain why $a = b = 0$ implies infinitely many solutions. Ignore any other possible infinitely many solution cases.

(a) $A\vec{x} = \vec{B}$ has a unique solution $\Leftrightarrow |A| \neq 0$.

$$\begin{pmatrix} c-a & -1 & 0 & b \\ a-c & 1 & a & b^2-b \\ 4a-2c & 3 & a & b^2 \end{pmatrix} \begin{matrix} \text{Combo}(1,2,1) \\ \text{Combo}(1,3,3) \end{matrix} \rightarrow \begin{pmatrix} c-a & -1 & 0 & b \\ 0 & 0 & a & b^2 \\ c+a & 0 & a & b^2+3b \end{pmatrix}$$

Then $|A| = (-1)(-1) \begin{vmatrix} c+a & a \\ a & a \end{vmatrix} = -a(c+a)$. Unique sol $\Leftrightarrow a(c+a) \neq 0$

(b) Then the last frame above is

$$\begin{pmatrix} c & -1 & 0 & b \\ 0 & 0 & 0 & b^2 \\ c & 0 & 0 & b^2+3b \end{pmatrix}$$

The 2nd equation is $0 = b^2$, a signal eq, meaning no sol for $a=0$ and $b \neq 0$

(c) Then the last frame is

$$\begin{pmatrix} c & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 \end{pmatrix}$$

A free variable is z , because it does not appear in the system. Zero is always a sol, so the system is consistent. Therefore, it has ∞ -many solutions.

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2 (Vector Spaces) Do all parts. Details not required for (a)-(d)

- (a) [10%] True or false. There is a subspace S of \mathcal{R}^3 containing none of the vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.
 $S = \text{span}(\vec{0})$ contains none of them.
- (b) [10%] True or false: The set of solutions \vec{x} in \mathcal{R}^3 of a consistent matrix equation $A\vec{x} = \vec{b}$ can equal all vectors in \mathcal{R}^3 . Let $A = \text{zero matrix}$, $\vec{b} = \text{zero vector}$.
- (c) [10%] True or false: Relations $xy = 0$, $y + z = 0$ define a subspace in \mathcal{R}^3 . See below
- (d) [10%] True or false: Equations $x + y = 0$, $y + z = 0$ define a subspace in \mathcal{R}^3 . See below
- (e) [20%] Two linear algebra theorems are able to conclude that the set S of all linear combinations of the functions $\sin(2x)$, e^x , $\cosh(2x)$ is a vector space of functions. State the the two theorems
- (f) [40%] Find a basis of vectors for the subspace of \mathcal{R}^4 given by the system of restriction equations

$$\begin{aligned} 3x_1 + 10x_2 + 2x_3 + 4x_4 &= 0, \\ 2x_1 + 4x_2 + x_3 + 2x_4 &= 0, \\ -2x_1 + x_2 &= 0, \\ 2x_1 + 12x_2 + 2x_3 + 4x_4 &= 0 \end{aligned}$$

- (c) False, by not a subspace theorem: $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ satisfy the relations but their sum $= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ fails to satisfy $xy = 0$.
- (d) True, by the kernel theorem, because the relations are homogeneous linear algebraic equations of vector-matrix form $A\vec{x} = \vec{0}$.
- (e) Theorem 1. The span of a set of vectors in a vector space V is a subspace S of V . (span theorem)
- Theorem 2. If S satisfies 1, 2, 3 below, then S is a subspace of vector space V . (subspace criterion)
1. $\vec{0}$ is in S ; 2. \vec{x}, \vec{y} in $S \Rightarrow \vec{x} + \vec{y}$ in S ; 3. \vec{x} in $S, c = \text{scalar} \Rightarrow c\vec{x}$ in S .

(f) The augmented matrix has rref

$$\left(\begin{array}{cccc|c} 1 & 0 & 1/4 & 1/2 & 0 \\ 0 & 1 & 1/8 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \rightarrow \begin{cases} x_1 + \frac{1}{4}x_3 + \frac{1}{2}x_4 = 0 \\ x_2 + \frac{1}{8}x_3 + \frac{1}{4}x_4 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases} \rightarrow \vec{x} = \begin{pmatrix} -\frac{1}{4}x_3 - \frac{1}{2}x_4 \\ -\frac{1}{8}x_3 - \frac{1}{4}x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

Then

$$\text{Subspace} = \text{Span} \left(\frac{\partial \vec{x}}{\partial x_3}, \frac{\partial \vec{x}}{\partial x_4} \right) = \text{Span} \left(\begin{pmatrix} -1/4 \\ -1/8 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/4 \\ 0 \\ 1 \end{pmatrix} \right)$$

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$$\text{Basis} = \frac{\partial \vec{x}}{\partial x_3}, \frac{\partial \vec{x}}{\partial x_4} = \left(\begin{pmatrix} -1/4 \\ -1/8 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/4 \\ 0 \\ 1 \end{pmatrix} \right)$$

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3 (Independence and Dependence) Do all parts

- (a) [10%] State an independence test for 3 vectors in \mathbb{R}^4 . Write the hypothesis and conclusion, not just the name of the test.
- (b) [10%] State another [different than (a)] independence test for 3 vectors in \mathbb{R}^4 .
- (c) [10%] For any matrix A , $\text{nullity}(A)$ equals the number of free variables for the problem $Ax = \vec{0}$. How many pivot columns in a 10×10 matrix A with $\text{nullity}(A) = 3$?
- (d) [30%] Let v_1, v_2, v_3, v_4 denote the rows of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & -2 & 0 & -6 & 0 \\ 0 & 2 & 0 & 5 & 1 \end{pmatrix}$$

Decide if the four rows v_1, v_2, v_3, v_4 are independent and display the details of the chosen independence test.

- (e) [40%] Extract from the list below a largest set of independent vectors.

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 3 \\ 3 \\ -1 \\ 0 \\ 5 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 2 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_5 = \begin{pmatrix} 0 \\ 3 \\ 3 \\ 3 \\ 0 \\ 9 \end{pmatrix}, \vec{v}_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}$$

- (a) Three vectors in \mathbb{R}^4 are independent \Leftrightarrow the augmented matrix A of the three vectors has rank = 3.
- (b) • Solve $c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$ for c_1, c_2, c_3 . The vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent $\Leftrightarrow c_1 = c_2 = c_3 = 0$.
- A determinant test: Let B be derived from A in part (a) by removing one row. If $|B| \neq 0$, then the vectors are independent. [However, this test is not \Leftrightarrow .]
- (c) Rank + nullity = # vars = 10, so rank = 7. The number of pivots = rank = 7.
- (d) • Add rows 1, 2 to get row 4. The rows are dependent.
• Or, compute $|A| = 0 \Leftrightarrow$ dependence of the rows.
- (e) Form the augmented matrix of \vec{v}_1 to \vec{v}_6 :

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 & 3 & 0 \\ 0 & 3 & 2 & 1 & 3 & 0 \\ 0 & -1 & -2 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 1 & 9 & 2 \end{pmatrix}, \text{The rref}(A) = \begin{pmatrix} 0 & 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -3 & -\frac{3}{2} \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix}, \text{pivots} = 2, 3.$$

Largest indep set
= $\{\vec{v}_2, \vec{v}_3\}$.

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