4. (Linear Equations)

(a) [50%] Solve the linear model. Show all integrating factor steps.

\[
\begin{align*}
3x'(t) &= -48 + \frac{12}{2t+5}x(t), \\
x(0) &= 40
\end{align*}
\]

(b) [20%] Solve the homogeneous equation \( \frac{dy}{dx} - (2x - 1)y = 0 \).

(c) [30%] Solve \( 13 \frac{dy}{dx} + 26y = \frac{7}{2} \) using the superposition principle \( y = y_h + y_p \). Expected are answers for \( y_h \) and \( y_p \).

\[\begin{align*}
\text{(a)} & \quad \chi' - \frac{4}{2t+5} \chi = -\frac{48}{3} \Rightarrow (\chi W) = \frac{-16}{W} \\
& \quad \int W \, dt = e^{-2 \ln (2t+5) + c}, \quad \text{choose } W = (2t+5)^{-2}. \text{ Then} \\
& \quad \chi = c (2t+5)^{-2} + 8 (2t+5)^{-1}, \quad \text{Because } \chi(0) = 40, \quad \text{Then} \\
& \quad 40 = c/25 + 40 \Rightarrow c = 0 \text{ and} \quad \chi = 16t + 40
\end{align*}\]

\[\begin{align*}
\text{(b) The answer is } & \quad y = \frac{c}{\text{integrating factor}} = \frac{c}{e^{-x^2+x}} \\
\text{(c) Choose } y_p: & \quad 0 + 26y = \frac{7}{2} \Rightarrow y_p = \frac{7}{52} \\
& \quad \text{Compute } y_h = \frac{c}{\text{integrating factor}} = \frac{c}{e^{2x}} \Rightarrow y_h = \frac{c}{e^{2x}} \\
& \quad y = y_h + y_p
\end{align*}\]
5. (Stability)
   (a) [50%] Draw a phase line diagram for the differential equation
   \[
   \frac{dx}{dt} = \sinh(x) \left( 2 - |2x - 4| \right)^3 (2 + x)(x^2 - 4)(1 - x^2)^3
   \]
   Expected in the phase line diagram are equilibrium points and signs of \(dx/dt\). Definition:
   \(\sinh(x) = \frac{1}{2} (e^x - e^{-x})\)
   \(\sinh(x) = 0 \iff x = 0\); \(2 - |2x - 4| = 0 \iff 2x - 4 = \pm 2 \implies x = 3 \text{ or } x = 1\).
   Other roots are \(-2, -1, 2, 1, -1\). There are 6 roots.

   \[+ - + - + - + \]
   \[-2 -1 0 1 1 2 1 3 \]

   (b) [50%] Assume an autonomous equation \(x'(t) = f(x(t))\). Draw a phase diagram with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate. funnel, spout, node [neither spout nor funnel], stable, unstable.

   \[+ + - - + + + + \]
   \[-8 -4 -2 0 1 \]

   Use this page to start your solution. Attach extra pages as needed.