Differential Equations and Linear Algebra 2250
Midterm Exam 1
Version 2, 16 Feb 2012

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Quadrature Equations)
   (a) [25%] Solve \( y' = \frac{1-x}{1+x} \).
   (b) [25%] Solve \( y' = (\cos x + 2)(\cos x - 2) \).
   (c) [25%] Solve \( y' = x^2 \cos(x^3) \), \( y(0) = 1 \).
   (d) [25%] Find the position \( x(t) \) from the velocity model \( \frac{dx}{dt} = e^{3t} v(t) \), \( v(0) = 0 \) and the position model \( \frac{dx}{dt} = v(t) \), \( x(0) = \frac{394}{3} \).

   \[ y' = \frac{1-x}{1+x} \Rightarrow y = -x + 2 \ln(1+x) + c \]

   \[ y' = \cos^2 x - 4 = \frac{1 + \cos 2x}{2} \Rightarrow y = -\frac{7}{2} x + \frac{1}{4} \sin(2x) + c \]

   \[ y' = \frac{1}{3} (3x^2) \cos(x^3) = \frac{1}{3} \cos(u) du \text{ where } u = x^3. \text{ Then} \]
   \[ y = \frac{1}{3} \sin u + c = \frac{1}{3} \sin(x^3) + c. \text{ Because } y(0) = 1, \text{ then } c = 1, \]
   \[ y = 1 + \frac{1}{3} \sin(x^3) \]

   By quadrature, \( e^{3t} v = 4 e^{3t} + c \). Then \( v(0) = 0 \) implies \( c = -4 \).

   We solved for \( v(t) = 4 - 4e^{-3t} \). Now solve
   \[
   \begin{cases}
   x' = 4 - 4e^{-3t} \\
   x(0) = \frac{394}{3}
   \end{cases}
   \]

   By quadrature,
   \[ x = 4t + \frac{4}{3} e^{-3t} + c \]
   \[ \frac{394}{3} = 0 + \frac{4}{3} + c \]
   \[ 100 = c \]
   \[ x(t) = 4t + \frac{4}{3} e^{-3t} + 100 \]

Use this page to start your solution. Attach extra pages as needed.
2. (Classification of Equations)
The differential equation $y' = f(x, y)$ is defined to be separable provided $f(x, y) = F(x)G(y)$ for some functions $F$ and $G$.

(a) [40%] Check (X) the problems that can be converted into separable form. No details expected.

- $y' + xy = y(y + x) - x^2y^2$
- $y' = (x - 1)(y + 1) + y^2$
- $y' = \cos(x + y)$
- $e^xy' = y + x^2$

(b) [10%] State a partial derivative test that decides if $y' = f(x, y)$ is a quadrature differential equation.

(c) [20%] Apply classification tests to show that $y' = x + y^2$ is not a linear differential equation. Supply all details.

(d) [30%] Apply a test to show that $y' = e^y + \tan|x|$ is not separable. Supply all details.

\[ y' + xy = y^2 + xy - x^2y^2 \Rightarrow y' = y^2 - x^2y^2 = (1 - x^2)y^2 = F \cdot G \]
\[ y' = (x - 1)(y + 1) + y^2 \Rightarrow y' = xy - y + x - 1 + y^2 \quad \text{not separable} \]
\[ \frac{fx}{f} = \frac{y + 1}{xy - y + x - 1 + y^2} \quad \text{Substitute } x = 1, \quad \frac{fx}{f} = \frac{y + 1}{y^2} \quad \text{depends on } y \]
\[ y' = \cos(x + y) \quad \text{not separable} \]
\[ \frac{fx}{f} = \frac{-\sin(x + y)}{\cos(x + y)} = -\tan(x + y) \quad \text{depends on } y \]
\[ e^xy' = y + x^2 \quad \text{not separable} \]
\[ \frac{fx}{f} = \frac{e^x}{(y + x^2)e^x} = \frac{1}{y + x^2} \quad \text{depends on } x \]

(b) $y' = f(x, y)$ is quadrature $\Rightarrow \frac{df}{dy} = 0$

(c) $\frac{fx}{f} = \frac{1}{x + y^2} \quad \text{depends on } y \Rightarrow \text{not separable (not requested)}$
\[ \frac{fy}{f} = 2y \quad \text{depends on } y \Rightarrow \text{not linear DE} \]

(d) Let $f(x, y) = e^y + \tan|x|$. For $x > 0$, $x = 0$, $x < 0$,
\[ \frac{fy}{f} = \frac{e^y}{e^y + \tan|x|} \quad \text{Substitute } y = 0, \tan \frac{fy}{f} = \frac{1}{1 + \tan|x|} \quad \text{depends on } x \]

Use this page to start your solution. Attach extra pages as needed.
3. (Solve a Separable Equation)

Given \((xy + y)y' = ((x + 1) \sin(x) + x)(y + 2)(y - 2)\).

Find a non-equilibrium solution in implicit form.

To save time, do not solve for \(y\) explicitly and do not solve for equilibrium solutions.

\[
\begin{align*}
\text{Separated form} & \quad \frac{y'}{G} = F \\
F & = \sin x + \frac{x}{1 + x} - 1 + \frac{1}{4x} \\
G & = \frac{(y+2)(y-2)}{y}
\end{align*}
\]

\[
\int F \, dx = -\cos x + x - \ln|1+x| + C_1
\]

\[
\int \frac{y'}{G} \, dx = \int \frac{yy'dx}{(y+2)(y-2)}
\]
\[
= \int \left(\frac{A}{y+2} + \frac{B}{y-2}\right) y' \, dx
\]
\[
= \frac{1}{2} \ln|y+2| + \frac{1}{2} \ln|y-2| + C_2
\]

\[
\int \frac{y'}{G} \, dx = \int F \, dx
\]
\[
\Rightarrow \quad \frac{1}{2} \ln|y+2| + \frac{1}{2} \ln|y-2| = -\cos x + x - \ln|1+x| + C
\]