Name ____

Differential Equations and Linear Algebra 2250

Midterm Exam 3

Version 1, Thu 12 April 2012

Scores
1.
2.

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Chapter 10)** Complete all parts. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(1a) [40%] Display the details of Laplace's method to solve the system for $y(t)$. Don't waste time solving for $x(t)$!

$$\begin{aligned} x' &= 4x, \\ y' &= x + 3y, \\ x(0) &= 1, \quad y(0) = 2. \end{aligned}$$

Answer:

The Laplace resolvent equation $(sI - A)\mathcal{L}(\mathbf{u}) = \mathbf{u}(0)$ can be written out to find a 2×2 linear system for unknowns $\mathcal{L}(x(t))$, $\mathcal{L}(y(t))$:

$$(s-4)\mathcal{L}(x) + (0)\mathcal{L}(y) = 1, \quad (-1)\mathcal{L}(x) + (s-3)\mathcal{L}(y) = 2.$$

Eimination or Cramer's rule applies to this system to solve for $\mathcal{L}(y(t)) = \dfrac{1}{s-4} + \dfrac{1}{s-3}$. Then the backward table implies $y(t) = e^{4t} + e^{3t}$. The unrequested answer is $x(t) = e^{4t}$.

(1b) [30%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{7s^2 + 6s + 3}{s^2(s-1)^2}.$$

Answer:

$\mathcal{L}(f(t)) = \frac{3}{s^2} + \frac{16}{(s-1)^2} + \frac{12}{s} + \frac{-12}{s-1} = \mathcal{L}(3t + 16te^t + 12 - 12e^t)$ implies $f(t) = 3t + 16te^t + 12 - 12e^t$.

(1c) [30%] Solve for $f(t)$, given

$$-\frac{d}{ds}\mathcal{L}(f(t)) + 2\frac{d^2}{ds^2}\mathcal{L}(tf(t)) = \frac{36}{(s+1)^4}.$$

Answer:

Use the s-differentiation theorem, shift theorem and the backward Laplace table to get $\mathcal{L}(tf(t)) + 2\mathcal{L}((-t)^2 tf(t)) = 36\mathcal{L}(t^3e^{-t}/6)$. Lerch's theorem implies $(t + 2t^3)f(t) = 6t^3e^{-t}$. Then $f(t) = 2t^3e^{-t}/(t + 2t^3)$.

Use this page to start your solution. Attach extra pages as needed.

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2. **(Chapter 10)** Complete all parts.

(2a) [60%] Fill in the blank spaces in the Laplace table:

Forward Table

$f(t)$	$\mathcal{L}(f(t))$
t^3	$\dfrac{6}{s^4}$
$e^{3t}\sin(2t)$	
$t^2 e^{-t/3}$	
$te^{-t}\sin(5t)$	

Backward Table

$\mathcal{L}(f(t))$	$f(t)$
$\dfrac{3}{s^2+9}$	$\sin 3t$
$\dfrac{s-2}{s^2-8s+17}$	
$\dfrac{4}{(2s+3)^2}$	
$\dfrac{s}{s^2+6s+18}$	

Answer:

Forward: $\dfrac{2}{(s-3)^2+4}$, $\dfrac{54}{(3s+1)^3}$, $-\dfrac{d}{ds}\dfrac{5}{s^2+25}\Big|_{s\to s+1} = \dfrac{10(s+1)}{((s+1)^2+25)^2}$.

Backward: $e^{4t}\cos(t) + 2e^{4t}\sin(t)$, $te^{-3t/2}$, $e^{-3t}\cos(3t) - e^{-3t}\sin(3t)$.

(2b) [40%] Find $\mathcal{L}(x(t))$, given $x(t) = t\mathbf{u}(t-2) + e^{t-1}\mathbf{u}(t-1)$, where $\mathbf{u}$ is the unit step function defined by $\mathbf{u}(t) = 1$ for $t \ge 0$, $\mathbf{u}(t) = 0$ for $t < 0$.

Answer:

Use the second shifting theorem

$$\mathcal{L}(f(t-a)\mathbf{u}(t-a)) = e^{-as}\mathcal{L}(f(t)).$$

Write $x(t) = (t-2)\mathbf{u}(t-2) + 2\mathbf{u}(t-2) + e^{t-1}\mathbf{u}(t-1)$. Then $\mathcal{L}(x(t)) = \mathcal{L}((t-2)\mathbf{u}(t-2)) + 2\mathcal{L}(\mathbf{u}(t-2)) + \mathcal{L}(e^{t-1}\mathbf{u}(t-1)) = e^{-2s}\mathcal{L}(t) + 2e^{-2s}\mathcal{L}(1) + e^{-s}\mathcal{L}(e^t) = e^{-2s}\left(\dfrac{1}{s^2} + \dfrac{2}{s}\right) + e^{-s}\dfrac{1}{s-1}$.

Use this page to start your solution. Attach extra pages as needed.