EXAMPLE:
A Non-Oscillating Second-Order System

The Problem

The figure shows two plates which can slide horizontally relative to one another. We wish to find a mathematical model for the motions of each of the two plates, solve it, and verify non-oscillation from the model.

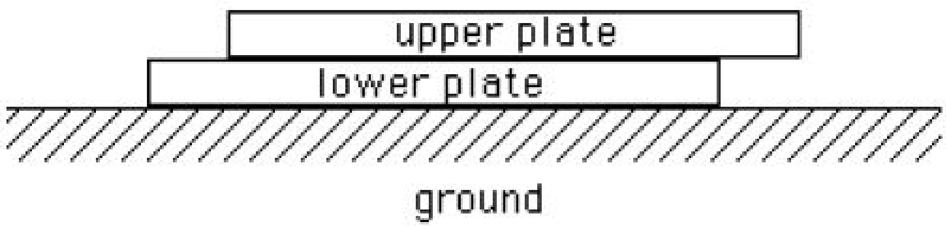

Figure. **A mechanical system with two movable plates**.

Assumptions and Notation

- The plates are rigid.
- The plates have masses $\boldsymbol{m_1}$, $\boldsymbol{m_2}$.
- Only horizontal motion is considered. Then each plate may be represented by an ideal inertia.
- The plate velocities will be denoted $\boldsymbol{v_1}$, $\boldsymbol{v_2}$.
- Assume two sources of energy dissipation: (1) linear friction $\boldsymbol{b_1}$(velocity) due to the motion of the top plate relative to the bottom and (2) linear friction $\boldsymbol{b_2}$(velocity) due to the motion of the bottom plate relative to ground.

Differential Equation Model

$$\begin{aligned} \boldsymbol{m_1 \frac{dv_1}{dt}} &= \boldsymbol{-(b_1 v_1 + b_2(v_1 - v_2))} \\ \boldsymbol{m_2 \frac{dv_2}{dt}} &= \boldsymbol{b_2(v_1 - v_2)} \end{aligned}$$

Derivation Detail.

Velocity $\boldsymbol{v_3 = v_1 - v_2}$ is the relative velocity between the two plates. It appears twice in the above equations, which are just Newton's second law on the left and the sum of the frictional forces on the right. Signs are determined by physics-style vector diagrams, or by mechanical engineering bond graphs.

Matrix Differential Equation Models

The **velocity model** is a translation of the scalar equations above into a first order matrix differential system of the form

$$\boldsymbol{M x'(t) = B x(t).}$$

Symbol $\boldsymbol{M}$ is the mass matrix and $\boldsymbol{B}$ is the coefficient matrix.

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -b_1 - b_2 & b_2 \\ b_2 & -b_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The **distance model** translates via equations $v_1 = dx_1/dt$, $v_2 = dx_2/dt$ into the second order matrix differential system

$$\frac{d^2}{dt^2} \begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{b_1}{m_1} - \frac{b_2}{m_1} & \frac{b_2}{m_1} \\ \\ \frac{b_2}{m_2} & -\frac{b_2}{m_2} \end{pmatrix} \begin{pmatrix} x_1' \\ \\ x_2' \end{pmatrix}$$

which has the symbolic form

$$\boldsymbol{x'' = A x.}$$

Distance Model Solution and Non-Oscillation

The model can be solved explicitly, using a computer algebra system like Mathematica or Maple. Here's the Maple code:

```
de1:=diff(x1(t),t,t)=-(b1+b2)/m1*diff(x1(t),t)+b2/m1*diff(x2(t),t);
de2:=diff(x2(t),t,t)=b2/m2*diff(x1(t),t)-b2/m2*diff(x2(t),t);
dsolve({de1,de2},{x1(t),x2(t)});
```

The answer is especially complicated, so we remove the mystery by choosing $b_1/m_1 = 1$ and $b_2/m_2 = 2$, which gives an answer similar to all others possible. Below, $\lambda_1 = (-5+\sqrt{17})/2$, $\lambda_2 = -(5+\sqrt{17})/2$, which are both negative numbers.

$$\begin{aligned} x_1(t) &= C_2 + C_3 e^{\lambda_1 t} + C_4 e^{\lambda_2 t} \\ x_2(t) &= \frac{1}{4}(1+\sqrt{17})C_3 e^{\lambda_1 t} + \frac{1}{4}(1-\sqrt{17})C_4 e^{\lambda_2 t} + \frac{3}{2}C_2 + C_1 \end{aligned}$$

Both $x_1(t)$, $x_2(t)$ have a constant limit at $t = \infty$, therefore they don't oscillate. A typical plot of $x_1(t)$, $x_2(t)$ appears in the next figure.

Distance Model Component Plot

In the figure, the components have been specialized to

$$x_1(t) = 1 + e^{\left(-5+\sqrt{17}\right)t/2} + e^{-\left(5+\sqrt{17}\right)t/2},$$

$$x_2(t) = \frac{1}{4}(1+\sqrt{17})\, e^{\left(-5+\sqrt{17}\right)t/2} + \frac{1}{4}(1-\sqrt{17})\, e^{-\left(5+\sqrt{17}\right)t/2} + 5/2.$$

These equations are the result of choosing all constants C_1 to C_4 equal to 1. The plot shows the non-oscillation of the components x_1, x_2.

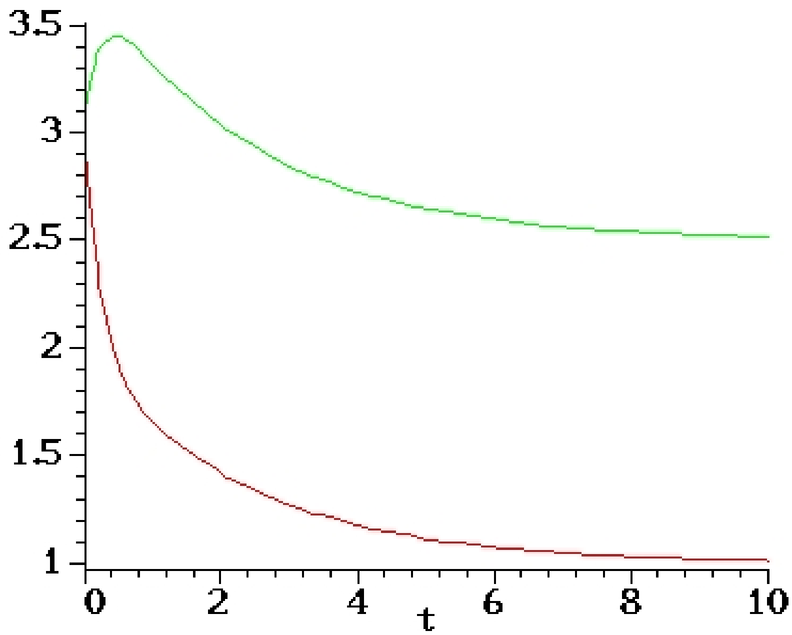

Figure. **Component plot for the movable plate problem**.