## Geometry of linear transformations

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|---|---|---|
| $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ | Scaling | Sub-classes **Dilation** ($k > 1$) and **Contraction** ($0 < k < 1$). |
| $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ | Projection | Define $\mathrm{proj}_L(\mathrm{x}) = (\mathrm{x} \cdot \mathrm{u})\mathrm{u}$ where $\mathrm{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a unit vector, $u_1^2 + u_2^2 = 1$. The matrix is $\begin{pmatrix} u_1 u_1 & u_1 u_2 \\ u_1 u_2 & u_2 u_2 \end{pmatrix}$ |
| $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ | Reflection | Define $\mathrm{refl}_L(\mathrm{x}) = 2(\mathrm{x} \cdot \mathrm{u})\mathrm{u} - \mathrm{x}$. The matrix is $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, $a^2 + b^2 = 1$. |
| $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ | Rotation | In general, $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ |
| $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ | Scaled Rotation | In general, $\begin{pmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$ |
| $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ | Vertical Shear | Change vertical $y \to y + kx$, leave $x$ fixed. |
| $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ | Horizontal Shear | Change horizontal $x \to x + ky$, leave $y$ fixed. |

## Properties of Geometric Transformations

- The columns of a projection matrix are scalar multiples of a single unit vector $\mathbf{u}$, therefore the columns are either the zero vector or else a vector parallel to $\mathbf{u}$.
- The columns of a reflection matrix are unit vectors that are pairwise orthogonal, that is, their pairwise dot products are zero.
- A shear can be classified as horizontal or vertical by its effect in mapping columns of the identity matrix. A horizontal shear leaves the first column of the identity matrix fixed, whereas a vertical shear leaves the second column of the identity matrix fixed.