# Elementary Matrices and Frame Sequences

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**Elementary Matrices**

**Definition**. An elementary matrix $\boldsymbol{E}$ is the result of applying a combination, multiply or swap rule to the identity matrix.

An elementary matrix is then the **second frame** after a `combo`, `swap` or `mult` toolkit operation which has been applied to a **first frame** equal to the identity matrix.

**Example**:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

First frame = identity matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$$

Second frame
Elementary combo matrix
`combo(1,3,-5)`

## Computer algebra systems and elementary matrices

The computer algebra system `maple` displays typical $4 \times 4$ elementary matrices (C=Combination, M=Multiply, S=Swap) as follows.

| | |
|---|---|
| `with(linalg):` | `with(LinearAlgebra):` |
| `Id:=diag(1,1,1,1);` | `Id:=IdentityMatrix(4);` |
| `C:=addrow(Id,2,3,c);` | `C:=RowOperation(Id,[3,2],c);` |
| `M:=mulrow(Id,3,m);` | `M:=RowOperation(Id,3,m);` |
| `S:=swaprow(Id,1,4);` | `S:=RowOperation(Id,[4,1]);` |

The answers:

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \boldsymbol{c} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boldsymbol{m} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{S} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

## Constructing elementary matrices $E$ and their inverses $E^{-1}$

| | |
|---|---|
| **Mult** | Change a one in the identity matrix to symbol $m \neq 0$. |
| **Combo** | Change a zero in the identity matrix to symbol $c$. |
| **Swap** | Interchange two rows of the identity matrix. |

### Constructing $E^{-1}$ from elementary matrix $E$

| | |
|---|---|
| **Mult** | Change diagonal multiplier $m \neq 0$ in $E$ to $1/m$. |
| **Combo** | Change multiplier $c$ in $E$ to $-c$. |
| **Swap** | The inverse of $E$ is $E$ itself. |

## Fundamental Theorem on Elementary Matrices

**Theorem 1 (Frame sequences and elementary matrices)**
In a frame sequence, let the second frame $\boldsymbol{A_2}$ be obtained from the first frame $\boldsymbol{A_1}$ by a `combo`, `swap` or `mult` toolkit operation. Let $\boldsymbol{n}$ equal the row dimenson of $\boldsymbol{A_1}$.Then there is correspondingly an $\boldsymbol{n \times n}$ `combo`, `swap` or `mult` elementary matrix $\boldsymbol{E}$ such that

$$\boldsymbol{A_2 = E A_1.}$$

**Theorem 2 (The $\mathbf{rref}$ and elementary matrices)**
Let $\boldsymbol{A}$ be a given matrix of row dimension $\boldsymbol{n}$. Then there exist $\boldsymbol{n \times n}$ elementary matrices $\boldsymbol{E_1}$, $\boldsymbol{E_2}$, ..., $\boldsymbol{E_k}$ such that

$$\mathbf{rref}(\boldsymbol{A}) = \boldsymbol{E_k \cdots E_2 E_1 A.}$$

**Proof of Theorem 1**

The first result is the observation that left multiplication of matrix $\boldsymbol{A}_1$ by elementary matrix $\boldsymbol{E}$ gives the answer $\boldsymbol{A}_2 = \boldsymbol{E}\boldsymbol{A}_1$ which is obtained by applying the corresponding `combo`, `swap` or `mult` toolkit operation. This fact is discovered by doing examples, then a formal proof can be constructed (not presented here).

**Proof of Theorem 2**

The second result applies the first result multiple times to obtain elementary matrices $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, ... which represent the multiply, combination and swap operations performed in the frame sequence which take the First Frame $\boldsymbol{A}_1 = \boldsymbol{A}$ into the Last Frame $\boldsymbol{A}_{k+1} = \mathbf{rref}(\boldsymbol{A}_1)$. Combining the identities

$$\boldsymbol{A}_2 = \boldsymbol{E}_1\boldsymbol{A}_1, \quad \boldsymbol{A}_3 = \boldsymbol{E}_2\boldsymbol{A}_2, \quad \ldots, \quad \boldsymbol{A}_{k+1} = \boldsymbol{E}_k\boldsymbol{A}_k$$

gives the matrix multiply equation

$$\boldsymbol{A}_{k+1} = \boldsymbol{E}_k\boldsymbol{E}_{k-1}\cdots\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A}_1$$

or equivalently the theorem's result, because $\boldsymbol{A}_{k+1} = \mathbf{rref}(\boldsymbol{A})$ and $\boldsymbol{A}_1 = \boldsymbol{A}$.

## A certain 6-frame sequence ________________.

$$A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 6 & 3 \end{pmatrix}$$ Frame 1, original matrix.

$$A_2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -6 \\ 3 & 6 & 3 \end{pmatrix}$$ Frame 2, combo(1,2,-2).

$$A_3 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 3 & 6 & 3 \end{pmatrix}$$ Frame 3, mult(2,-1/6).

$$A_4 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -6 \end{pmatrix}$$ Frame 4, combo(1,3,-3).

$$A_5 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ Frame 5, combo(2,3,-6).

$$A_6 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ Frame 6, combo(2,1,-3). Found $\mathrm{rref}(A_1)$.

**Continued**

The corresponding $3 \times 3$ elementary matrices are

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Frame 2, combo(1,2,-2) applied to $I$.

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Frame 3, mult(2,-1/6) applied to $I$.

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$ Frame 4, combo(1,3,-3) applied to $I$.

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix}$$ Frame 5, combo(2,3,-6) applied to $I$.

$$E_5 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Frame 6, combo(2,1,-3) applied to $I$.

**Frame Sequence Details**

| | |
|---|---|
| $\boldsymbol{A_2 = E_1 A_1}$ | Frame 2, $\boldsymbol{E_1}$ equals combo(1,2,-2) on $\boldsymbol{I}$. |
| $\boldsymbol{A_3 = E_2 A_2}$ | Frame 3, $\boldsymbol{E_2}$ equals mult(2,-1/6) on $\boldsymbol{I}$. |
| $\boldsymbol{A_4 = E_3 A_3}$ | Frame 4, $\boldsymbol{E_3}$ equals combo(1,3,-3) on $\boldsymbol{I}$. |
| $\boldsymbol{A_5 = E_4 A_4}$ | Frame 5, $\boldsymbol{E_4}$ equals combo(2,3,-6) on $\boldsymbol{I}$. |
| $\boldsymbol{A_6 = E_5 A_5}$ | Frame 6, $\boldsymbol{E_5}$ equals combo(2,1,-3) on $\boldsymbol{I}$. |
| $\boldsymbol{A_6 = E_5 E_4 E_3 E_2 E_1 A_1}$ | Summary frames 1-6. |

Then

$$\mathbf{rref}(\boldsymbol{A_1}) = \boldsymbol{E_5 E_4 E_3 E_2 E_1 A_1},$$

which is the result of the Theorem.

## Fundamental Theorem Illustrated

The summary:

$$A_6 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A_1$$

Because $A_6 = \mathrm{rref}(A_1)$, the above equation gives the inverse relationship

$$A_1 = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1}\, \mathrm{rref}(A_1).$$

Each inverse matrix is simplified by the rules for constructing $E^{-1}$ from elementary matrix $E$, the result being

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathrm{rref}(A_1)$$

## Theorem 3 (RREF Inverse Method)

$$\mathbf{rref}(\mathbf{aug}(\boldsymbol{A}, \boldsymbol{I})) = \mathbf{aug}(\boldsymbol{I}, \boldsymbol{B}) \quad \text{if and only if} \quad \boldsymbol{AB} = \boldsymbol{I}.$$

**Proof**: For *any* matrix $\boldsymbol{E}$ there is the matrix multiply identity

$$\boldsymbol{E}\,\mathbf{aug}(\boldsymbol{C}, \boldsymbol{D}) = \mathbf{aug}(\boldsymbol{EC}, \boldsymbol{ED}).$$

This identity is proved by arguing that each side has identical columns. For example, $\mathbf{col}(\text{LHS}, 1) = \boldsymbol{E}\,\mathbf{col}(\boldsymbol{C}, 1) = \mathbf{col}(\text{RHS}, 1)$.

Assume $\boldsymbol{C} = \mathbf{aug}(\boldsymbol{A}, \boldsymbol{I})$ satisfies $\mathbf{rref}(\boldsymbol{C}) = \mathbf{aug}(\boldsymbol{I}, \boldsymbol{B})$. The fundamental theorem of elementary matrices implies $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{C} = \mathbf{rref}(\boldsymbol{C})$. Then

$$\mathbf{rref}(\boldsymbol{C}) = \mathbf{aug}(\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{A}, \boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{I}) = \mathbf{aug}(\boldsymbol{I}, \boldsymbol{B})$$

implies that $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I}$ and $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{I} = \boldsymbol{B}$. Together, $\boldsymbol{BA} = \boldsymbol{I}$ and then $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$.

Conversely, assume that $\boldsymbol{AB} = \boldsymbol{I}$. Then $\boldsymbol{A}$ has inverse $\boldsymbol{B}$. The fundamental theorem of elementary matrices implies the identity $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{A} = \mathbf{rref}(\boldsymbol{A}) = \boldsymbol{I}$. It follows that $\boldsymbol{B} = \boldsymbol{E}_k \cdots \boldsymbol{E}_1$. Then $\mathbf{rref}(\boldsymbol{C}) = \boldsymbol{E}_k \cdots \boldsymbol{E}_1\,\mathbf{aug}(\boldsymbol{A}, \boldsymbol{I}) = \mathbf{aug}(\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{A}, \boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{I}) = \mathbf{aug}(\boldsymbol{I}, \boldsymbol{B})$.