1 Heaviside's Method

This practical method, popularized by the English electrical engineer Oliver Heaviside (1850–1925)., systematically converts a polynomial quotient

$$\frac{a_0 + a_1 s + \cdots + a_n s^n}{b_0 + b_1 s + \cdots + b_m s^m} \tag{1}$$

into a sum of **partial fractions** (defined below). It is assumed that $a_0, .., a_n, b_0, \ldots, b_m$ are constants and the polynomial quotient (1) has limit zero at $s = \infty$.

1.1 Partial Fraction Theory

Definition. A fraction with a constant in the numerator and a denominator having just one root is called a **partial fraction**. Such fractions have the form

$$\frac{A}{(s - s_0)^k}. \tag{2}$$

The numerator in (2) is a real or complex constant A and the denominator has exactly one root $s = s_0$.
In college algebra, it is shown that a rational function (1) can be expressed as the sum of **partial fractions**. The power $(s - s_0)^k$ in (2) **must divide the denominator** in (1).
Assume fraction (1) has **real coefficients**.
Real Root. If s_0 in (2) is real, then A is *real*. In the partial fraction expansion of (1), such a fraction will appear if and only if $(s - s_0)^k$ is a factor of the denominator $b_0 + b_1 s + \cdots + b_m s^m$.
Non-Real Root. If $s_0 = \alpha + i\beta$ in (2) is *complex*, then $(s - \overline{s_0})^k$ also divides the denominator in (1), where $\overline{s_0} = \alpha - i\beta$ is the complex conjugate of s_0. The corresponding partial fractions used in the partial fraction expansion of (1) turn out to be complex conjugates of one another, which can be paired and re-written as a fraction

$$\frac{A}{(s - s_0)^k} + \frac{\overline{A}}{(s - \overline{s_0})^k} = \frac{Q(s)}{((s - \alpha)^2 + \beta^2)^k}, \tag{3}$$

where $Q(s)$ is a *real* polynomial. This justifies the replacement of all partial fractions $A/(s - s_0)^k$ with complex s_0 by

$$\frac{B + Cs}{((s - s_0)(s - \overline{s_0}))^k} = \frac{B + Cs}{((s - \alpha)^2 + \beta^2)^k},$$

in which B and C are real constants. This **real form** is preferred over the complex fractions on the left, because integral tables typically contain only real formulas.

Simple Roots. Assume that (1) has *real coefficients* and the denominator of the fraction (1) has **distinct real roots** $s_1, \ldots, s_N$ and **distinct complex roots** $\alpha_1 \pm i\beta_1, \ldots, \alpha_M \pm i\beta_M$. The partial fraction expansion of (1) is a sum given in terms of *real* constants A_p, B_q, C_q by

$$\frac{a_0 + a_1 s + \cdots + a_n s^n}{b_0 + b_1 s + \cdots + b_m s^m} = \sum_{p=1}^{N} \frac{A_p}{s - s_p} + \sum_{q=1}^{M} \frac{B_q + C_q(s - \alpha_q)}{(s - \alpha_q)^2 + \beta_q^2} . \tag{4}$$

Multiple Roots. Assume (1) has *real coefficients* and the denominator of the fraction (1) has possibly **multiple roots**. Let N_p be the multiplicity of real root s_p and let M_q be the multiplicity of complex root $\alpha_q + i\beta_q$ ($\beta_q > 0$), $1 \le p \le N$, $1 \le q \le M$. The partial fraction expansion of (1) is given in terms of *real* constants $A_{p,k}$, $B_{q,k}$, $C_{q,k}$ by

$$\sum_{p=1}^{N} \sum_{1 \le k \le N_p} \frac{A_{p,k}}{(s - s_p)^k} + \sum_{q=1}^{M} \sum_{1 \le k \le M_q} \frac{B_{q,k} + C_{q,k}(s - \alpha_q)}{((s - \alpha_q)^2 + \beta_q^2)^k} . \tag{5}$$

Summary. The theory for simple roots and multiple roots can be distilled as follows.

> A polynomial quotient p/q with limit zero at infinity has a unique expansion into partial fractions. A partial fraction is either a constant divided by a divisor of q having exactly one root, or else a linear function divided by a real divisor of q, having exactly one complex conjugate pair of roots.

1.2 A Failsafe Method

Consider the expansion in partial fractions

$$\frac{s - 1}{s(s+1)^2(s^2+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{Ds + E}{s^2 + 1}. \tag{6}$$

The five undetermined real constants A through E are found by **clearing the fractions**, that is, multiply (6) by the denominator on the left to obtain the polynomial equation

$$\begin{aligned} s - 1 &= A(s+1)^2(s^2+1) + Bs(s+1)(s^2+1) \\ &\quad + Cs(s^2+1) + (Ds + E)s(s+1)^2. \end{aligned} \tag{7}$$

Next, five different values of s are substituted into (7) to obtain equations for the five unknowns A through E. We always use the **roots of the denominator** to start: $s = 0$, $s = -1$, $s = i$, $s = -i$ are the roots of $s(s+1)^2(s^2+1) = 0$. Each complex root results in two equations, by taking real and imaginary parts. The complex conjugate root $s = -i$ is not used, because it duplicates equations already obtained from $s = i$. The three roots $s = 0$, $s = -1$, $s = i$ give only four equations, so we invent another value $s = 1$ to get the fifth equation:

$$\begin{array}{rcll} -1 &=& A & (s=0) \\ -2 &=& -2C - 2(-D+E) & (s=-1) \\ i-1 &=& (Di+E)i(i+1)^2 & (s=i) \\ 0 &=& 8A+4B+2C+4(D+E) & (s=1) \end{array} \tag{8}$$

Because D and E are real, the complex equation $(s = i)$ becomes two equations, as follows.

$i - 1 = (Di + E)i(i^2 + 2i + 1)$	Expand power.
$i - 1 = -2Di - 2E$	Simplify using $i^2 = -1$.
$1 = -2D$	Equate imaginary parts.
$-1 = -2E$	Equate real parts.

Solving the 5×5 system, the answers are $A = -1$, $B = 2$, $C = 0$, $D = -1/2$, $E = 1/2$.

1.3 Heaviside's Coverup Method

The method applies only to the case of distinct roots of the denominator in (1). Extensions to multiple-root cases can be made; see page 4.
To illustrate Oliver Heaviside's ideas, consider the problem details

$$\frac{2s+1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} \tag{9}$$

Equation (9) uses college algebra partial fractions.

[**.** mysterious details]Mysterious Details Oliver Heaviside proposed to find in (9) the constant $C = -\frac{1}{2}$ by a **cover–up method**:

$$\left.\frac{2s+1}{s(s-1)\boxed{}}\right|_{\boxed{s+1}=0} = \frac{C}{\boxed{}}.$$

The *instructions* are to cover–up the matching factors $(s + 1)$ on the left and right with box $\boxed{}$ (Heaviside used two fingertips), then evaluate on the left at the *root* s which causes the box contents to be zero. The other terms on the right are replaced by zero.
To justify Heaviside's cover–up method, **clear the fraction** $C/(s + 1)$, that is, multiply (9) by the denominator $\boxed{s+1}$ of the partial fraction $C/(s + 1)$ to obtain the *partially-cleared fraction relation*

$$\frac{(2s+1)\boxed{(s+1)}}{s(s-1)\boxed{(s+1)}} = \frac{A\boxed{(s+1)}}{s} + \frac{B\boxed{(s+1)}}{s-1} + \frac{C\boxed{(s+1)}}{\boxed{(s+1)}}.$$

Set $\boxed{(s+1)} = 0$ in the display. Cancellations left and right plus annihilation of two terms on the right gives Heaviside's prescription

$$\left.\frac{2s+1}{s(s-1)}\right|_{s+1=0} = C.$$

The factor $(s+1)$ in (9) is by no means special: the same procedure applies to find A and B. The method works for denominators with simple roots, that is, no repeated roots are allowed.

Heaviside's method in words:

> To determine A in a given partial fraction $\frac{A}{s-s_0}$, multiply the relation by $(s-s_0)$, which partially clears the fraction. Substitute for s via equation $s-s_0=0$.

Extension to Multiple Roots. Heaviside's method can be extended to the case of repeated roots. The basic idea is to *factor–out the repeats*. To illustrate, consider the partial fraction expansion details

$R = \dfrac{1}{(s+1)^2(s+2)}$	A sample rational function having repeated roots.
$= \dfrac{1}{s+1}\left(\dfrac{1}{(s+1)(s+2)}\right)$	Factor–out the repeats.
$= \dfrac{1}{s+1}\left(\dfrac{1}{s+1} + \dfrac{-1}{s+2}\right)$	Apply the cover–up method to the simple root fraction.
$= \dfrac{1}{(s+1)^2} + \dfrac{-1}{(s+1)(s+2)}$	Multiply.
$= \dfrac{1}{(s+1)^2} + \dfrac{-1}{s+1} + \dfrac{1}{s+2}$	Apply the cover–up method to the last fraction on the right.

Terms with only one root in the denominator are already partial fractions. Thus the work centers on expansion of quotients in which the denominator has two or more roots.

Special Methods. Heaviside's method has a useful extension for the case of roots of multiplicity two. To illustrate, consider these details:

$R = \dfrac{1}{(s+1)^2(s+2)}$	$\boxed{1}$ A fraction with multiple roots.
$= \dfrac{A}{s+1} + \dfrac{B}{(s+1)^2} + \dfrac{C}{s+2}$	$\boxed{2}$ See equation (5), page 2.
$= \dfrac{A}{s+1} + \dfrac{1}{(s+1)^2} + \dfrac{1}{s+2}$	$\boxed{3}$ Find B and C by Heaviside's cover–up method.

$$= \frac{-1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s+2} \qquad \boxed{4}\ \text{Details below.}$$

We discuss $\boxed{4}$ details. Multiply the equation $\boxed{1} = \boxed{2}$ by $s+1$ to partially clear fractions, the same step as the cover-up method:

$$\frac{1}{(s+1)(s+2)} = A + \frac{B}{s+1} + \frac{C(s+1)}{s+2}.$$

We don't substitute $s+1=0$, because it gives infinity for the second term. Instead, set $s=\infty$ to get the equation $0 = A + C$. Because $C = 1$ from $\boxed{3}$, then $A = -1$.

The illustration works for one root of multiplicity two, because $s = \infty$ will resolve the coefficient not found by the cover–up method.

In general, if the denominator in (1) has a root s_0 of multiplicity k, then the partial fraction expansion contains terms

$$\frac{A_1}{s-s_0} + \frac{A_2}{(s-s_0)^2} + \cdots + \frac{A_k}{(s-s_0)^k}.$$

Heaviside's cover–up method directly finds A_k, but not A_1 to A_{k-1}.

Cover-up Method and Complex Numbers. Consider the partial fraction expansion

$$\frac{10}{(s+1)(s^2+9)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+9}.$$

The symbols A, B, C are real. The value of A can be found directly by the cover-up method, giving $A = 1$. To find B and C, multiply the fraction expansion by s^2+9, in order to partially clear fractions, then formally set $s^2+9=0$ to obtain the two equations

$$\frac{10}{s+1} = Bs + C, \quad s^2 + 9 = 0.$$

The method applies the identical idea used for one real root. By clearing fractions in the first, the equations become

$$10 = Bs^2 + Cs + Bs + C, \quad s^2 + 9 = 0.$$

Substitute $s^2 = -9$ into the first equation to give the linear equation

$$10 = (-9B + C) + (B + C)s.$$

Because this linear equation has two complex roots $s = \pm 3i$, then real constants B, C satisfy the 2×2 system

$$\begin{array}{rcrcr} -9B & + & C & = & 10, \\ B & + & C & = & 0. \end{array}$$

Solving gives $B = -1$, $C = 1$.

The same method applies especially to fractions with 3-term denominators, like s^2+s+1. The only change made in the details is the replacement $s^2 \to -s-1$. By repeated application of $s^2 = -s-1$, the first equation can be distilled into one linear equation in s with two roots. As before, a 2×2 system results.

1.4 Examples

Example 1.1 (Partial Fractions I) *Show the details of the partial fraction expansion*

$$\frac{s^3+2s^2+2s+5}{(s-1)(s^2+4)(s^2+2s+2)} = \frac{2/5}{s-1} + \frac{1/2}{s^2+4} - \frac{1}{10}\frac{7+4\,s}{s^2+2\,s+2}.$$

Solution:
Background. The problem originates a partial fraction equality, which occurs as one step in Laplace theory.
College algebra detail. College algebra partial fractions theory says that there exist real constants A, B, C, D, E satisfying the identity

$$\frac{s^3+2s^2+2s+5}{(s-1)(s^2+4)(s^2+2s+2)} = \frac{A}{s-1} + \frac{B+Cs}{s^2+4} + \frac{D+Es}{s^2+2\,s+2}.$$

As explained on page 1, the complex conjugate roots $\pm 2i$ and $-1\pm i$ are not represented as terms $c/(s-s_0)$, but in the combined real form seen in the above display, which is suited for use with integral tables.
The **failsafe method** applies to find the constants. In this method, the fractions are cleared to obtain the polynomial relation

$$\begin{aligned} s^3+2s^2+2s+5 &= A(s^2+4)(s^2+2s+2) \\ &\quad +(B+Cs)(s-1)(s^2+2s+2) \\ &\quad +(D+Es)(s-1)(s^2+4). \end{aligned}$$

The roots of the denominator $(s-1)(s^2+4)(s^2+2s+2)$ to be inserted into the previous equation are $s=1$, $s=2i$, $s=-1+i$. The conjugate roots $s=-2i$ and $s=-1-i$ are not used. Each complex root generates two equations, by equating real and imaginary parts, therefore there will be 5 equations in 5 unknowns. Substitution of $s=1$, $s=2i$, $s=-1+i$ gives three equations

$$\begin{array}{lrcl} s=1 & 10 &=& 25A, \\ s=2i & -4i-3 &=& (B+2iC)(2i-1)(-4+4i+2), \\ s=-1+i & 5 &=& (D-E+Ei)(-2+i)(2-2(-1+i)). \end{array}$$

Writing each expanded complex equation in terms of its real and imaginary parts, explained in detail below, gives 5 equations

$$\begin{array}{lrcrcr} s=1 & 2 &=& 5A, && \\ s=2i & -3 &=& -6B &+& 16C, \\ s=2i & -4 &=& -8B &-& 12C, \\ s=-1+i & 5 &=& -6D &-& 2E, \\ s=-1+i & 0 &=& 8D &-& 14E. \end{array}$$

The equations are solved to give $A=2/5$, $B=1/2$, $C=0$, $D=-7/10$, $E=-2/5$ (details for B, C below).
Complex equation to two real equations. It is an algebraic mystery how exactly the complex equation

$$-4i-3 = (B+2iC)(2i-1)(-4+4i+2)$$

gets converted into two real equations. The process is explained here.
First, the complex equation is expanded, as though it is a polynomial in variable i, to give the steps

$$\begin{array}{rcll} -4i-3 & = & (B+2iC)(2i-1)(-2+4i) & \\ & = & (B+2iC)(-4i+2+8i^2-4i) & \text{Expand.} \\ & = & (B+2iC)(-6-8i) & \text{Use } i^2=-1. \\ & = & -6B-12iC-8Bi+16C & \text{Expand, use } i^2=-1. \\ & = & (-6B+16C)+(-8B-12C)i & \text{Convert to form } x+yi. \end{array}$$

Next, the two sides are compared. Because B and C are real, then the real part of the right side is $(-6B+16C)$ and the imaginary part of the right side is $(-8B-12C)$. Equating matching parts on each side gives the equations

$$\begin{array}{rcl} -6B+16C & = & -3, \\ -8B-12C & = & -4, \end{array}$$

which is a 2×2 linear system for the unknowns B, C.
Solving the 2×2 system. Such a system with a unique solution can be solved by Cramer's rule, matrix inversion or elimination. The answer: $B=1/2$, $C=0$.
The easiest method turns out to be elimination. Multiply the first equation by 4 and the second equation by 3, then subtract to obtain $C=0$. Then the first equation is $-6B+0=-3$, implying $B=1/2$.

Example 1.2 (Partial Fractions II) *Verify the partial fraction expansion*

$$\begin{array}{rcl} \dfrac{s^5+8\,s^4+23\,s^3+37\,s^2+29\,s+10}{(s+1)^2\,(s^2+s+1)^2} & = & \dfrac{1}{s+1}+\dfrac{2}{(s+1)^2} \\ & & +\dfrac{3}{s^2+s+1} \\ & & +\dfrac{4+5\,s}{(s^2+s+1)^2} \end{array}$$

Solution:
Basic partial fraction theory implies that there are real constants a, b, c, d, e, f satisfying the equation

$$\begin{array}{rcl} \dfrac{s^5+8\,s^4+23\,s^3+37\,s^2+29\,s+10}{(s+1)^2\,(s^2+s+1)^2} & = & \dfrac{a}{s+1}+\dfrac{b}{(s+1)^2} \\ & & +\dfrac{c+ds}{s^2+s+1}+\dfrac{e+f\,s}{(s^2+s+1)^2} \end{array} \tag{10}$$

The **failsafe** method applies to clear fractions and replace the fractional equation by the polynomial relation

$$\begin{array}{rcl} s^5+8\,s^4+23\,s^3+37\,s^2+29\,s+10 & = & a(s+1)(s^2+s+1)^2 \\ & & +b(s^2+s+1)^2 \\ & & +(c+ds)(s^2+s+1)(s+1)^2 \\ & & +(e+f\,s)(s+1)^2 \end{array}$$

However, the prognosis for the resultant algebra is grime: only three of the six required equations can be obtained by substitution of the roots ($s=-1$, $s=-1/2+i\sqrt{3}/2$) of

the denominator. We abandon the idea, because of the complexity of the 6×6 system of linear equations required to solve for a through f.
Instead, the fraction on the left of (10) is written with repeated roots factored out, as follows:

$$\frac{1}{(s+1)(s^2+s+1)} \left(\frac{p(x)}{(s+1)(s^2+s+1)} \right),$$
$$p(x) = s^5 + 8\,s^4 + 23\,s^3 + 37\,s^2 + 29\,s + 10.$$

Long division gives the formula

$$\frac{p(x)}{(s+1)(s^2+s+1)} = s^2 + 6s + 9.$$

Therefore, the fraction on the left of (10) can be written as

$$\frac{p(x)}{(s+1)^2(s^2+s+1)^2} = \frac{(s+3)^2}{(s+1)(s^2+s+1)}.$$