Applied Linear Algebra 2270-3
Midterm Exam 1
Wednesday, 7 October, 2008

Instructions: This in-class exam is designed for 50 minutes. No tables, notes, books or calculators allowed.

1. (Inverse of a matrix) Supply details for one of these:

a. If A and B are $n \times n$ invertible, then $(AB)^{-1} = A^{-1}B^{-1}$ can be false.

b. If square matrices A and B satisfy $AB = I$, then $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for each vector $\mathbf{b}$.

Please staple this page to your solution. Write your initials on all pages.

2. (Elementary Matrices) Let A be a 3×3 matrix. Let

$$F = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Assume F is obtained from A by the following sequential row operations: (1) Swap rows 2 and 3; (2) Add -2 times row 2 to row 3; (3) Add 3 times row 1 to row 2; (4) Multiply row 2 by -3.

a. Write a matrix multiplication formula for F in terms of explicit elementary matrices and the matrix A. (80%)

b. Find A. (20%)

3. (RREF method)

Part I. State a theorem about non-homogeneous systems $A\mathbf{x} = \mathbf{b}$ which concludes that the $n \times n$ system has a unique solution. Then supply a proof of the theorem. [20%]

Part II. Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} 1 & 2b-c & -a \\ 1 & c & a \\ 2 & 2b & -a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ -a \\ -a \end{pmatrix}$$

(a). Determine those values of a, b and c such that the system has a unique solution. (40%)

(b). Determine those values of a, b and c such that the system has no solution. (20%)

(c). Determine those values of a, b and c such that the system has infinitely many solutions. (20%)

To save time, don't attempt to solve the equations or to produce a complete frame sequence.

4. (Matrix algebra)
Do one of these:

a. Let A be a 4×3 matrix and B a 3×4 matrix. Explain using matrix algebra and the three possibilities why $\mathbf{rref}(AB) = I$ cannot happen.

b. For $n \times n$ matrices A, B, C assume that $AB = C^2$, $AC = I$, $ABC = B$. Prove that $A^3B = I$.

5. (Geometry and linear transformations)

Part I. Classify $T(\mathbf{x}) = A\mathbf{x}$ geometrically as scaling, projection onto line L, reflection in line L, pure rotation by angle θ, rotation composed with scaling, horizontal shear, vertical shear. Define angle θ where applicable. [60%]

a. $A = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$

b. $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

c. $A = \frac{1}{4}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$

d. $A = \frac{1}{2}\begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}$

e. $A = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$

Part II. Give details. [40%]

f. Define reflection in a line L with unit direction $\mathbf{u}$.

g. Display the 3×3 matrix A of a projection onto a line L through $(0, 0, 0)$ in unit direction $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$.

h. Display the 2×2 matrix A of a planar rotation clockwise by angle θ.