100

Name Quincy Loertscher

Final 2270-2

Final Exam *Introduction to Linear Algebra* May 4, 2010.

Instructions. The exam is designed for 2 hours. Calculators, books, notes and computers are not allowed.

Ch3. (Subspaces of $\mathcal{R}^n$ and Their Dimensions) Do all.

[30%] **Ch3(a)**: Find a basis in $\mathcal{R}^4$ for the image of A.

$$A = \begin{array}{c} \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 4 & 7 \end{pmatrix} \\ \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \end{array}$$

[30%] **Ch3(b)**: Find a basis in $\mathcal{R}^4$ for the kernel of A.

$$A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 3 \\ 5 & 3 & -3 & 7 \end{pmatrix}$$

[40%] **Ch3(c)**: Assume the set V of all 3×3 matrices is a vector space with the usual addition and scalar multiplication. Let S be the subset of V defined by the relations

$$\text{All matrices of the form} \begin{pmatrix} a & b & 0 \\ a+b & c & a \\ a-b & a+c & 0 \end{pmatrix}$$

$$\text{with} \begin{cases} a+b+c &= 0, \\ a+2c &= 0. \end{cases}$$

Prove that S is a subspace of V.

$a+2c=0$
$a=-2c$

$a+b+c=0$
$-2c+b+c=0$
$b=c$

closed under addition means if the matrices D & E are in S, then so is (D+E).

closed under scalar mult means D & kD are in S.

matrix is of form $\begin{pmatrix} -2c & c & 0 \\ -c & c & -2c \\ -3c & -c & 0 \end{pmatrix}$

or $c\begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & -2 \\ -3 & -1 & 0 \end{pmatrix}$ since c is an arbitrary constant, S is closed under scalar multiplication.

Start your solution on this page. Please append to this page any additional sheets for this problem.

Ch3a. im A = span of its columns

$$A\vec{x} = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_m\vec{v}_m$$

So to get a basis in $\mathbb{R}^4$, we must eliminate the redundant (meaning linearly dependent) vectors.

$\begin{pmatrix} 0\\0\\0\\0 \end{pmatrix}$ is redundant $\begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} = 0(v_2)+0(v_3)+0(v_4)$

$\vec{v}_2 = \begin{pmatrix} 1\\0\\1\\3 \end{pmatrix}$; $\vec{v}_3$ is independent since $\begin{pmatrix} 1\\0\\1\\3 \end{pmatrix} \neq k\begin{pmatrix} 1\\1\\2\\4 \end{pmatrix}$ where k is an arbitrary constant

Test $\vec{v}_4$: $\begin{pmatrix} 2\\1\\3\\7 \end{pmatrix} = c_1\begin{pmatrix} 1\\0\\1\\3 \end{pmatrix} + c_2\begin{pmatrix} 1\\1\\2\\4 \end{pmatrix}$

$c_1 + c_2 = 2 \leftarrow c_2 = 1$ $\quad$ $c_1 + 2c_2 = 3$ $\quad$ $3c_1 + 4c_2 = 7$

$c_1 + 1 = 2$ $\quad$ $1 + 2 = 3$ $\quad$ $3 + 4 = 7$

$c_1 = 1$

$\vec{v}_4 = \vec{v}_2 + \vec{v}_3$, so it is linearly dependent

$$\boxed{\text{basis im}A = \begin{pmatrix} 1\\0\\1\\3 \end{pmatrix}, \begin{pmatrix} 1\\1\\2\\4 \end{pmatrix}}$$

Ch3b. $A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 3 \\ 5 & 3 & -3 & 7 \end{pmatrix}$ combo(1,2,−2) combo(1,3,−3) combo(1,4,−5) $\rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -2 & 2 & -3 \\ 0 & -2 & 2 & -3 \\ 0 & -2 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -2 & 2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ combo$(2,1,\frac{1}{2})$ $\rightarrow$

ker$A$ = $\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & -2 & 2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ mult$(2,-\frac{1}{2})$ $\rightarrow \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & -1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} x_1\\x_2\\x_3\\x_4 \end{pmatrix} = t_1\begin{pmatrix} -\frac{1}{2}\\ -\frac{3}{2}\\ 0\\ 1 \end{pmatrix} + t_2\begin{pmatrix} 0\\1\\1\\0 \end{pmatrix}$ take $\frac{\partial}{\partial t_1}$ & $\frac{\partial}{\partial t_2}$ for basis

basis ker$A$ = $\begin{pmatrix} -\frac{1}{2}\\ -\frac{3}{2}\\ 0\\ 1 \end{pmatrix}, \begin{pmatrix} 0\\1\\1\\0 \end{pmatrix}$

Ch3c. Conditions for subspace:
1. contains 0 element
2. Is closed under linear combinations (addition & scalar multiplications)

$C = A + B$

since $(a+b)$, a, and b are arbitrary constants, S is closed under addition and Condition 2 is met

S is a subspace of V

Condition 1: If $a=b=c=0$, we produce the zero matrix and $a+b+c=0$, $a+2c=0$ are satisfied.

Condition 2: Proved scalar mult on previous page. Since the matrix is of the form $c\begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & -2 \\ -3 & -1 & 0 \end{pmatrix}$, C can be split up into two arbitrary constants $(a+b)$

$$\underbrace{(a+b)\begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & -2 \\ -3 & -1 & 0 \end{pmatrix}}_{\text{matrix } C} = \underbrace{a\begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & -2 \\ -3 & -1 & 0 \end{pmatrix}}_{\text{matrix } A} + \underbrace{b\begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & -2 \\ -3 & -1 & 0 \end{pmatrix}}_{\text{matrix } B}$$

100

Name ______________________ **Final 2270-2**

Final Exam ***Introduction to Linear Algebra*** **May 4, 2010.**

Ch4. (Linear Spaces) Do all.

[40%] **Ch4(a)**: Let W be the vector space of all functions $f(x)$ defined on $-\infty < x < \infty$ with the usual rules for addition and scalar multiplication. Define V_1 to V_4 to be the subsets of W consisting of all polynomials $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3$, plus the condition given below.

Mark $\boxed{X}$ for the subsets V which are subspace of W. For the ones that fail, leave them marked $\boxed{}$, and explain your reason.

$\boxed{X}$ V_1: $\int_0^1 f(x)x dx = 0$, $f(-5) = f(5)$	$\boxed{X}$ V_2: $\int_0^1 f(x)x^2 dx = 0$, $f(0) = f(1)$
$\boxed{}$ V_3: $\int_0^1 f(x)x dx = f(-5)$, $f(1) > 0$	$\boxed{X}$ V_4: $\int_0^1 f(x)x dx = f(5) + f(-5)$

✓

[40%] **Ch4(b)**: Find bases for the image and kernel of T, defined by

$$T(\vec{x}) = \left(A^2 + 2A\right)\vec{x}, \quad A = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}.$$

[20%] **Ch4(c)**: Let V be a finite dimensional vector space. Define the terms **basis** and **dimension**.

V_1:

a. $\int_0^1 f(x)\cdot x\,dx = \int_0^1 c_0x + c_1x^2 + c_2x^3 + c_3x^4\,dx$

$= \frac{1}{2}c_0x^2 + \frac{1}{3}c_1x^3 + \frac{1}{4}c_2x^4 + \frac{1}{5}c_3x^5 \Big|_0^1$

$= \frac{1}{2}c_0 + \frac{1}{3}c_1 + \frac{1}{4}c_2 + \frac{1}{5}c_3 = 0$

$c_0 = \frac{-2}{3}c_1 - \frac{1}{2}c_2 - \frac{2}{5}c_3$

(Note: for $\int_0^1 f(x)x^2dx$, we only need to increase the denominator of each coefficient by 1)

$f(-5) = f(5)$

$c_0 - 5c_1 + 25c_2 - 125c_3 = c_0 + 5c_1 + 25c_2 + 125c_3$

$-5c_1 - 125c_3 = 5c_1 + 125c_3$

$10c_1 = 250c_3$

$c_1 = 25c_3$

$f(x) = -\frac{50}{3}c_3 + 25c_3x - \frac{1}{2}c_2 + c_2x^2 + c_3x^3$

$f(x) = c_2(x^2 - \frac{1}{2}) + c_3(x^3 + 25x - \frac{50}{3})$

V_1 contains 0 if $c_2 = c_3 = 0$

$kf(x)$ is in V_1 since kc_2 & kc_3 are also arbitrary constants and $f(x)+g(x)$ is in V_1

V_3: $\frac{1}{2}c_0 + \frac{1}{3}c_1 + \frac{1}{4}c_2 + \frac{1}{5}c_3 = c_0 - 5c_1 + 25c_2$

$\underbrace{c_0 + c_1 + c_2 + c_3 > 0}$

this means V_3 cannot contain the zero element, so V_3 fails to be a subspace.

V_2: $\cancel{c_0} = \cancel{c_0} + c_1 + c_2 + c_3$

$c_1 + c_2 + c_3 = 0$

$\frac{1}{3}c_0 + \frac{1}{4}c_1 + \frac{1}{5}c_2 + \frac{1}{6}c_3 = 0$

same process as V_1 only with slightly varied constants, so V_2 is a subspace

Start your solution on this page. Please append to this page any additional sheets for this problem.

4a cont. V_4 $\frac{1}{2}c_0 + \frac{1}{3}c_1 + \frac{1}{4}c_2 + \frac{1}{5}c_3 = c_0 + \ldots$

V_4 is a subspace by the same reasons as V_1, only the constants are varied.

4b. $T(\vec{x}) = B\vec{x}$ where $B = A^2 + 2A = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} + 2\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$

ker $B \rightarrow B\vec{x} = \vec{0}$

mult$(1, \frac{1}{2})$ combo $(1,2,-\frac{1}{2})$ $\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\frac{\partial}{\partial t_1}$ for basis

basis of ker $T = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$T(\vec{x}) = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$ so basis im $A = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

redundant

4c. The basis of a finite dimensional vector space is a set of vectors $\vec{v}_1, \ldots, \vec{v}_m$ that span V (any $\vec{x}$ in V can be expressed as $c_1\vec{v}_1 + \ldots + c_m\vec{v}_m$) and are linearly independent ($c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_m\vec{v}_m = \vec{0}$ only if $c_1 = c_2 = \ldots = c_m = 0$). The dimension of V is the number of linearly indepdendendent vectors we need to span V (m in this example).

Name ______________________

Final Exam *Introduction to Linear Algebra* May 4, 2010.

Ch5. (Orthogonality and Least Squares) Do all.

[30%] **Ch5(a)**: Find the orthogonal projection of

$$\vec{\mathbf{v}} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \quad \text{onto} \quad V = \textbf{span}\left(\underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{\vec{u}_1}, \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{\vec{u}_2} \right).$$

[10%] **Ch5(b)**: Define orthogonal complement $V^\perp$ for a subspace V of a vector space W.

[30%] **Ch5(c)**: Find the Gram-Schmidt orthonormal vectors $\vec{\mathbf{u}}_1$, $\vec{\mathbf{u}}_2$, $\vec{\mathbf{u}}_3$ for the subspace S of $\mathcal{R}^4$ spanned by the three vectors

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{v}}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

[10%] **Ch5(d)**: Display the normal equation in the theory of least squares.

[20%] **Ch5(e)**: Assume the columns of $m \times n$ matrix A are independent. Use the QR-decomposition formula to prove that the least squares normal equation can be written as $R\vec{\mathbf{x}}^* = Q^T\vec{\mathbf{b}}$.

5a. $\text{proj}_V \vec{v} = (\vec{v}\cdot\vec{u}_1)\vec{u}_1 + (\vec{v}\cdot\vec{u}_2)\vec{u}_2$ if $\vec{u}_1$ & $\vec{u}_2$ are orthonormal.

$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$; $\vec{u}_2 = \cancel{\tfrac{1}{\sqrt{2}}}\begin{pmatrix}-1\\1\\0\end{pmatrix}$ $\quad \text{proj}_V\vec{v} = \frac{1}{2}\Big[\underbrace{\begin{pmatrix}-1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}}_{1}\Big]\begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{1}{2}\Big[\underbrace{\begin{pmatrix}-1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix}}_{3}\Big]\begin{pmatrix}-1\\1\\0\end{pmatrix}$

$\vec{u}_1\cdot\vec{u}_2 = 0$ $\quad = \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{3}{2}\begin{pmatrix}-1\\1\\0\end{pmatrix}$

$\vec{u}_1\cdot\vec{u}_1 = 1$

$\vec{u}_2\cdot\vec{u}_2 = 1$

$\text{proj}_V\vec{v} = \begin{pmatrix}-1\\2\\0\end{pmatrix}$ $\qquad \boxed{\text{proj}_V(\vec{v}) = \begin{pmatrix}-1\\2\\0\end{pmatrix}}$

b. $V^\perp$ is the set of all vectors $\vec{x}$ such that $\vec{x}\cdot\vec{v} = 0$ for every $\vec{v}$ in V. By the Rank-Nullity Theorem: $\dim V^\perp + \dim V = \dim W$

c. $\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$ $\qquad \vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \vec{v}_2 - (\vec{u}_1\cdot\vec{v}_2)\vec{u}_1$

$= \begin{pmatrix}0\\0\\1\\0\end{pmatrix} - \frac{1}{2}\Big[\underbrace{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\\0\end{pmatrix}}_{1}\Big]\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\0\\\frac{1}{2}\\0\end{pmatrix}$

$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$ $\qquad \vec{v}_3^{\perp} = \vec{v}_3 - (\vec{u}_1\cdot\vec{v}_3)\vec{u}_1 - (\vec{u}_2\cdot\vec{v}_3)\vec{u}_2$

$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \begin{pmatrix}0\\-1\\0\\0\end{pmatrix}$ $\qquad \vec{v}_3^{\perp} = \begin{pmatrix}1\\-1\\1\\0\end{pmatrix} - \frac{1}{2}\Big[\underbrace{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}}_{2}\Big]\begin{pmatrix}1\\0\\1\\0\end{pmatrix} - \frac{1}{2}\Big[\underbrace{\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}}_{0}\Big]\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$

$= \begin{pmatrix}1\\-1\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\0\\0\end{pmatrix}$

Start your solution on this page. Please append to this page any additional sheets for this problem.

d. $A^T A \vec{x}^* = A^T \vec{b}$

e. ~~The columns of A are independent, so $\ker A = \{\vec{0}\} = \ker(A^T A)$, so $A^T A$ is invertible~~

$A = QR$ for QR factorization

$A^T = R^T Q^T$ Q is orthonormal $Q^{-1} = Q^T$

$A^T A = R^T Q^T Q R = R^T R$

R is an upper triangular matrix with diagonal elements $\begin{pmatrix} \|\vec{v}_1\| & & \text{stuff} \\ & \|\vec{v}_2^\perp\| & \\ & & \ddots \\ 0 & & \|\vec{v}_m^\perp\| \end{pmatrix}$ this is square

$$A^T A \vec{x} = A^T \vec{b}$$

$$(R^T)^{-1} \left(R^T R \vec{x}^* = R^T Q^T \vec{b} \right)$$

$$\boxed{R \vec{x}^* = Q^T \vec{b}}$$

since the columns of $A = (\vec{v}_1 \ldots \vec{v}_m)$ and they are independent, $\|\vec{v}_1\| \neq 0$ $\|\vec{v}_2^\perp\| \neq 0 \ldots \|\vec{v}_m^\perp\|$ so det R is defined and R is invertible (therefore its transpose is as well)

100

Final 2270-2

Name ______________________

Final Exam *Introduction to Linear Algebra* **May 4, 2010.**

Ch6. (Determinants) Do all.

[40%] **Ch6(a)**: This problem uses the adjugate identity $B\,\mathbf{adj}(B) = \mathbf{adj}(B)B = \det(B)I$.

Define the invertible matrix B and its adjugate (or adjoint) matrix C by the formulas

multiply

$$B = \begin{pmatrix} ? & ? & ? & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ ? & ? & ? & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 6 & 6 & 12 & 0 \\ -6 & -6 & ? & 0 \\ -3 & ? & 3 & ?\,x_2 \\ 2 & 2 & 4 & -6 \end{pmatrix}$$

x_1

$2x_2 = 0$

$x_2 = 0$

Symbol $\boxed{?}$ means the value of the entry does not affect the answer to this problem. Find the values of $\det(B)$, $\det(C)$ and $\det(2B^{-1}C(B^2)^T)$, where M^T means the transpose of matrix M.

[30%] **Ch6(b)**: Evaluate the determinant by any hybrid method.

$$\begin{vmatrix} 1 & 1 & 2 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & 1 & 2 & -3 \end{vmatrix}$$

[30%] **Ch6(c)**: Assume given 3×3 matrices A, B. Suppose $E_5E_4AB = E_3E_2E_1A$ and E_1, E_2, E_3, E_4, E_5 are elementary matrices representing respectively a a combination, a multiply by 4, a swap, a combination and a multiply by -2. Assume $\det(A) = -1$. Find $\det(5A^2B)$.

6a. $BC^T = \begin{pmatrix} \det B & & & \\ & \det B & & 0 \\ & & \det B & \\ 0 & & & \det B \end{pmatrix}$ $= x_1 x_2 - 18 = \det B$

$\boxed{\det B = -18}$

$2B = 2IB$

$\det(BC) = \det B \det C$ $\qquad BC = \det B\, I$

$\det(B)^4 = \det B \det C$ $\qquad \det(BC) = \det B^4$

$\boxed{\det C = \det(B)^3 = (-18)^3}$ $\qquad = \det B^2 = \det B \det B$

$\det(2B^{-1}C(B^2)^T) = \underbrace{\det(2B^{-1})}\det(C)\det(B^2)^T = \dfrac{\det B^5}{\det(2B)} = \dfrac{\det B^5}{\det(2I)\det B} = \dfrac{\det B^4}{\det 2I} = \dfrac{(-18)^4}{2^4}$

$\boxed{\det(2B^{-1}C(B^2)^T) = 9^4}$ $\qquad \det(B^{-1}) = \dfrac{1}{\det(B)}$

Start your solution on this page. Please append to this page any additional sheets for this problem.

1b. combos: $\det A = \det B$

mult: $\det B = C \det A$

swap: $\det B = -\det A$

$$\det A = (-1)^s \frac{\det \operatorname{rref} A}{c_1 c_2 c_3 \cdots c_m}$$

$$\det B = \det \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & 1 & 2 & -3 \end{pmatrix} \begin{matrix} \text{combo}(1,2,1) \\ \text{combo}(1,3,1) \\ \text{combo}(1,4,-1) \end{matrix}$$

$$= \det \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \text{combo}(2,3,-1)$$

$$= \det \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{matrix} \text{mult}(2,\tfrac{1}{3}) \\ \text{mult}(3,\tfrac{1}{3}) \\ \text{mult}(4,-\tfrac{1}{3}) \end{matrix}$$

$$= \frac{-1}{27} \det \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \text{goes away with combos} \\ \text{2 swap} \end{matrix}$$

$$= \frac{1}{27} \det I_4 = \frac{1}{27}$$

$$\det A = \frac{-1 \cdot 1}{\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{-1}{3}} = 27$$

$$\boxed{\det A = 27}$$

2c. $\det E_{combo} = 1$

$\det E_{swap} = -1$

$\det E_{mult} = m$

$\det E_1 = 1$

$\det E_2 = 4$

$\det E_3 = -1$

$\det E_4 = 1$

$\det E_5 = -2$

$$\det(E_5 E_4 A B = E_3 E_2 E_1 A)$$

$$\det E_5 \det E_4 \cancel{\det A} \det B = \det E_3 \det E_2 \det E_1 \cancel{\det A}$$

$$\det B = \frac{\det E_3 \det E_2 \det E_1}{\det E_4 \det E_5} = \frac{1(4)(-1)}{(1)(-2)} = 2$$

$$\det(5A^2B) = \det(5I) \det A \det A \det B$$

$$= 5^3 (-1)(-1)(2)$$

$$\boxed{\det(5A^2B) = 250}$$

100

Final 2270-2

Name ______________________

Final Exam *Introduction to Linear Algebra* May 4, 2010.

Ch7. (Eigenvalues and Eigenvectors) Do all.

[30%] **Ch7(a)**: Find the eigenvalues of

$$A = \begin{pmatrix} 2 & -3 & 6 & 44 \\ 3 & 2 & 11 & -5 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 2 & -3 \end{pmatrix}.$$

[10%] **Ch7(b)**: Suppose A is 10×10 and an eigenvector $\vec{\mathbf{v}}$ is known. What's the simplest way to find the corresponding eigenvalue?

[30%] **Ch7(c)**: A discrete dynamical system $\vec{\mathbf{x}}(n+1) = A\vec{\mathbf{x}}(n)$ has particular solution

$$\vec{\mathbf{x}}(n) = \begin{pmatrix} 2^n - 6^n \\ 2^n + 6^n \end{pmatrix}.$$

Find the matrix A.

[30%] **Ch7(d)**: Fourier's model for 2×2 matrices A is

$$A(c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2) = c_1\lambda_1\vec{\mathbf{v}}_1 + c_2\lambda_2\vec{\mathbf{v}}_2$$

where c_1, c_2 are arbitrary constants. Scalars λ_1, λ_2 and independent vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ depend only on A. Prove that Fourier's model is equivalent to the eigenpair equations

$$\begin{cases} A\vec{\mathbf{v}}_1 = \lambda_1\vec{\mathbf{v}}_1, \\ A\vec{\mathbf{v}}_2 = \lambda_2\vec{\mathbf{v}}_2 \end{cases}$$

with $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ independent.

7a. $A - \lambda I_4 = \begin{pmatrix} 2-\lambda & -3 & 6 & 44 \\ 3 & 2-\lambda & 11 & -5 \\ 0 & 0 & 3-\lambda & -4 \\ 0 & 0 & 2 & -3-\lambda \end{pmatrix}$

cofactor expansion by 1st column

$$\det(A-\lambda I_4) = (2-\lambda)\begin{vmatrix} 2-\lambda & 11 & -5 \\ 0 & 3-\lambda & -4 \\ 0 & 2 & -3-\lambda \end{vmatrix} - 3\begin{vmatrix} -3 & 6 & 44 \\ 0 & 3-\lambda & -4 \\ 0 & 2 & -3-\lambda \end{vmatrix}$$

$$= (2-\lambda)(2-\lambda)\begin{vmatrix} 3-\lambda & -4 \\ 2 & -3-\lambda \end{vmatrix} - 3(-3)\begin{vmatrix} 3-\lambda & -4 \\ 2 & -3-\lambda \end{vmatrix}$$

$$= [(2-\lambda)^2 + 9][(3-\lambda)(-3-\lambda) + 8] = 0$$

$$= [\lambda^2 - 4\lambda + 13][\lambda^2 - 1] = 0$$

$$\boxed{\lambda = 1, -1, 2+3i, 2-3i}$$

13
4
52

$6i$

$\lambda^2 - 4\lambda + 13 = 0$

$\lambda = \dfrac{4 \pm \sqrt{16 - 4(13)}}{2} = \dfrac{4 \pm \sqrt{16-52}}{2} = \dfrac{4 \pm \sqrt{-36}}{2}$

$\lambda = 2 \pm 3i$

Start your solution on this page. Please append to this page any additional sheets for this problem.

7b. Use the fact that $A\vec{v} = \lambda\vec{v}$ and solve by plugging in values of A & $\vec{v}$ (Of course, the simplest way is to just use a computer). ✓

7c. $\vec{x}(1) = A\vec{x}(0)$

$\vec{x}(2) = A\vec{x}(1) = A^2\vec{x}(0)$

$\vdots$

$\vec{x}(n) = A^n\vec{x}(0) = \lambda^n \vec{x}(0)$ eigenvalues

$\begin{pmatrix} 2^n - 6^n \\ 2^n + 6^n \end{pmatrix} = 2^n \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\text{eigenvectors}} - 6^n \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rightarrow A \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -6 \\ 2 & 6 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} a+b & b-a \\ c+d & d-c \end{pmatrix} = \begin{pmatrix} 2 & -6 \\ 2 & 6 \end{pmatrix}$

$a + b = 2 \quad b - a = -6$

$a = 2 - b \quad b - 2 + b = -6$

$a = 4 \quad 2b = -4$

$b = -2$

$c + d = 2 \quad d - c = 6$

$c = 2 - d \quad d - 2 + d = 6$

$c = -2 \quad 2d = 8$

$d = 4$

$A = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$

✓

7d. $\begin{cases} (A\vec{v}_1 = \lambda_1\vec{v}_1)\, c_1 \\ (A\vec{v}_2 = \lambda_2\vec{v}_2)\, c_2 \end{cases} \rightarrow \begin{cases} Ac_1\vec{v}_1 = \lambda_1 c_1\vec{v}_1 \\ Ac_2\vec{v}_2 = \lambda_2 c_2 \vec{v}_2 \end{cases}$ add equations together

$Ac_1\vec{v}_1 + Ac_2\vec{v}_2 = \lambda_1 c_1 \vec{v}_1 + \lambda_2 c_2 \vec{v}_2$ ✓

$A(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2$

Name ______________________ Final 2270-2

Final Exam ***Introduction to Linear Algebra*** **May 4, 2010.**

Ch8. (Symmetric Matrices, Singular Value Decomposition) Do all.

[50%] **Ch8(a):** Consider the symmetric matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

which has eigenvalues $-1, -1, 2$. Find an orthogonal matrix Q and a diagonal matrix D such that $AQ = QD$. The columns of Q form an orthonormal eigenbasis of A.

[50%] **Ch8(b):** Compute the three matrices U, Σ, V in the singular value decomposition $A = U\Sigma V^T$ for the 2×2 matrix $A = \begin{pmatrix} 6 & -7 \\ 2 & 6 \end{pmatrix}$.

a. $\lambda = -1 \rightarrow \begin{pmatrix} +1 & 1 & 1 \\ 1 & +1 & 1 \\ 1 & 1 & +1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow t_1 \overset{\vec{v}_1}{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}} + t_2 \overset{\vec{v}_2}{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$

$\lambda = 2 \rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow t_1 \overset{\vec{v}_3}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$

v_3 is already $\perp$ to $\vec{v}_1, \vec{v}_2$ since A is symmetric

$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $\qquad \vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$

$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ $\qquad = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\left[\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right]\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\qquad = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$\qquad = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

~~$Q = (\vec{u}_1\ \vec{u}_2\ \vec{u}_3)$ $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$~~

$$\boxed{Q = \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$$

Start your solution on this page. Please append to this page any additional sheets for this problem.

8b. $V = (\vec{v}_1\ \vec{v}_2)$ where $\vec{v}_1, \vec{v}_2$ are any orthonormal vectors (eigenvectors of A^TA)

$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$ where $\sigma_1 = \sqrt{\lambda_1}$ & $\sigma_2 = \sqrt{\lambda_2}$ where λ_1, λ_2 are the eigenvalues of A^TA

$U = (\vec{u}_1\ \vec{u}_2)$ where $\sigma_i \vec{u}_i = A\vec{v}_i$

$$A^TA = \begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix}\begin{pmatrix} 6 & -7 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 40 & -30 \\ -30 & 85 \end{pmatrix}$$

$$\det(A - \lambda I_2) = \begin{vmatrix} 40-\lambda & -30 \\ -30 & 85-\lambda \end{vmatrix} = (40-\lambda)(85-\lambda) - 900 = 0$$

$$\lambda^2 - 125\lambda + 3400 - 900 = \lambda^2 - 125\lambda + 2500 = 0$$

$$(\lambda - 100)(\lambda - 25) = 0 \qquad \lambda = 100, 25$$

$$\lambda = 100 \rightarrow \begin{pmatrix} -60 & -30 \\ -30 & -15 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow t_1\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\lambda = 25 \rightarrow \begin{pmatrix} 15 & -30 \\ -30 & 60 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \rightarrow t_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\vec{v}_1 = \tfrac{1}{\sqrt5}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \vec{v}_2 = \tfrac{1}{\sqrt5}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$A\vec{v}_1 = \sigma_1 \vec{u}_1$$

$$\begin{pmatrix} 6 & -7 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \sqrt{100}\,\vec{u}_1$$

$$\frac{1}{10}\begin{pmatrix} -20 \\ 10 \end{pmatrix} = \vec{u}_1$$

$$\vec{u}_1 = \tfrac{1}{\sqrt5}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$A\vec{v}_2 = \sigma_2 \vec{u}_2$$

$$\begin{pmatrix} 6 & -7 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \sqrt{25}\,\vec{u}_2$$

$$\frac{1}{5\sqrt5}\begin{pmatrix} 5 \\ 10 \end{pmatrix} = \vec{u}_2 = \frac{1}{\sqrt5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\boxed{U = \begin{pmatrix} \frac{-2}{\sqrt5} & \frac{1}{\sqrt5} \\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{pmatrix} \quad \Sigma = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix} \quad V = \begin{pmatrix} \frac{-1}{\sqrt5} & \frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \end{pmatrix}}$$

$$A = U\Sigma V^T$$

$$A = \tfrac{1}{5}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \cancel{10}^{\,2} & 0 \\ 0 & \cancel{5}^{\,1} \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 6 & -7 \\ 2 & 6 \end{pmatrix}$$

✓