Name ________________________ **2270-2**

Introduction to Linear Algebra 2270-2
Sample Midterm Exam 2 Spring 2010
Exam Date: 22 April

Instructions. This exam is designed for 50 minutes. Calculators, books, notes and computers are not allowed.

1. (Kernel and Image) Do three.

(a) Prove that $\mathbf{im}(A^TB^T) = \mathbf{ker}(BA)^\perp$, when matrix product BA is defined.

(b) It is known that each pivot column j of A satisfies the relation $E\,\mathbf{col}(A,j) = \mathbf{col}(I,j)$, for some invertible matrix E. Identify E from the theory of elementary matrices and prove the relation.

(c) Prove or disprove: $AB = I$ with A, B non-square implies $\mathbf{ker}(B) = \{\mathbf{0}\}$.

Additional problem types, for practise

(d) Suppose that $\mathbf{ker}(A) = \{\mathbf{0}\}$. Prove that $\mathbf{ker}(A^TA) = \{\mathbf{0}\}$.

(e) Suppose $\mathbf{ker}(A)$ and $\mathbf{ker}(B)$ are isomorphic. Are A and B similar?

(f) Prove or disprove: $\mathbf{ker}(\mathbf{rref}(BA)) = \mathbf{ker}(A)$, for all invertible matrices B.

(g) Prove or disprove: $\mathbf{ker}(\mathbf{rref}(A)) = \mathbf{ker}(A)$.

(h) Suppose A and B are both $n \times n$ of rank n. Prove or give a counterexample: the column spaces of A and B are identical.

(i) Illustrate the relation $\mathbf{rref}(A) = E_k \cdots E_2E_1A$ by a frame sequence and explicit elementary matrices for the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

(j) Prove or disprove: $\mathbf{ker}(B) = \mathbf{ker}(A)$, for all frames A and B in any frame sequence.

(k) Prove or disprove: If $B = S^{-1}AS$ for an invertible square matrix S, then $\mathbf{ker}(A) = \mathbf{ker}(B)$.

Start your solution on this page. Please append to this page any additional sheets for this problem.

2. (Vector Spaces, Subspaces, Independence and Bases) Do three.

(a) Let A be a 15×11 matrix. Prove or give a counterexample: $\dim(\mathbf{ker}(A)) + \dim(\mathbf{im}(A)) = 11$.

(b) Let V be the vector space of all polynomials $p(x) = c_0 + c_1 x + c_2 x^2$ under function addition and scalar multiplication. Let S be the subspace of V satisfying the relation $\int_{-1}^{1} p(x)dx = p(1)$. Find $\dim(S)$ and display a basis for S.

(c) Let $S = \left\{ \begin{pmatrix} a & b \\ -a & 2b \end{pmatrix} : a, b \ \text{real} \right\}$. Find a basis for S.

Additional problem types, for practise.

(d) Prove that vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ orthonormal in $\mathcal{R}^n$ are linearly independent.

(e) Prove that the column positions of leading ones in $\mathbf{rref}(A)$ identify independent columns of A.

(f) Let V be the vector space of all functions defined on the real line, using the usual definitions of function addition and scalar multiplication. Let $V_1 = \mathbf{span}\,\{1, x, 1-x, e^x\}$ and let S be the subset of all functions $f(x)$ in V_1 such that $f(0) = f(1)$. Prove that S is a subspace of V.

(g) Show that the set of all $m \times n$ matrices A which have exactly one element equal to 1, and all other elements zero, form a basis for the vector space of all $m \times n$ matrices.

(h) Let V be the vector space of all polynomials under function addition and scalar multiplication. Prove that $1, x, \ldots, x^n$ are independent in V.

(i) Let A be a 4×4 matrix. Suppose that, for any possible independent set $\mathbf{v}_1, \ldots, \mathbf{v}_4$, the set $A\mathbf{v}_1, \ldots, A\mathbf{v}_4$ is independent. Prove or give a counterexample: $\mathbf{ker}(A) = \{\mathbf{0}\}$.

(j) Let V be the vector space of all polynomials $c_0 + c_1 x + c_2 x^2$ under function addition and scalar multiplication. Prove that $1 - x$, $2x$, $(x-1)^2$ form a basis of V.

(k) Let V be the vector space of all 2×4 matrices A. Display a basis of V in which each basis element has exactly two nonzero entries. Include a proof that the displayed set is indeed a basis.

(l) Let V be the vector space of all functions defined on the real line, using the usual definitions of function addition and scalar multiplication. Let S be the set of all polynomials of degree less than 5 (e.g., $x^4 \in V$ but $x^5 \notin V$) that have zero constant term. Prove that S is a subspace of V.

Start your solution on this page. Please append to this page any additional sheets for this problem.

3. (Orthogonality, Gram-Schmidt and Least Squares) Complete three.

(a) Find the orthogonal projection of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ onto $V = \textbf{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}$.

(b) Find the QR-factorization of $A = \begin{pmatrix} 1 & 0 \\ 7 & 7 \\ 1 & 2 \end{pmatrix}$.

(c) Fit $c_0 + c_1 x$ to the data points $(0, 2)$, $(1, 0)$, $(2, 1)$, $(3, 1)$ using least squares. Sketch the solution and the data points as an answer check. This is a 2×2 system problem, and should take only 1-3 minutes to complete.

Additional problem types, for practise.

(d) For an inconsistent system $A\mathbf{x} = \mathbf{b}$, the least squares solutions $\mathbf{x}$ are the exact solutions of the normal equation. Define the normal equation and display the unique solution $\mathbf{x} = \mathbf{x}^*$ when $\textbf{ker}(A) = \{\mathbf{0}\}$.

(e) State the *near point theorem*. Then explain how the near point to $\mathbf{x}$ can equal $\mathbf{x}$ itself.

(f) Prove that the span of the Gram-Schmidt vectors $\mathbf{u}_1$, $\mathbf{u}_2$ equals exactly the span of the independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$ used to construct them.

(g) Prove that the product AB of two orthogonal matrices A and B is again orthogonal.

(h) Prove that an orthogonal matrix A has an inverse A^{-1} which is also orthogonal.