Quincy Loertscher, Azur Azapagic, Jiantao Bian,
Christian Diaz

Name ______________________ 2270-2

3. (Orthogonality, Gram-Schmidt and Least Squares) Complete three.

(a) Find the orthogonal projection of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ onto $V = \textbf{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}\right\}$.

(b) Find the QR-factorization of $A = \begin{pmatrix}1 & 0\\7 & 7\\1 & 2\end{pmatrix}$.

(c) Fit $c_0 + c_1 x$ to the data points $(0, 2)$, $(1, 0)$, $(2, 1)$, $(3, 1)$ using least squares. Sketch the solution and the data points as an answer check. This is a 2×2 system problem, and should take only 1-3 minutes to complete.

Additional problem types, for practise.

(d) For an inconsistent system $A\mathbf{x} = \mathbf{b}$, the least squares solutions $\mathbf{x}$ are the exact solutions of the normal equation. Define the normal equation and display the unique solution $\mathbf{x} = \mathbf{x}^*$ when $\mathbf{ker}(A) = \{\mathbf{0}\}$.

(e) State the *near point theorem*. Then explain how the near point to $\mathbf{x}$ can equal $\mathbf{x}$ itself.

(f) Prove that the span of the Gram-Schmidt vectors $\mathbf{u}_1$, $\mathbf{u}_2$ equals exactly the span of the independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$ used to construct them.

(g) Prove that the product AB of two orthogonal matrices A and B is again orthogonal.

(h) Prove that an orthogonal matrix A has an inverse A^{-1} which is also orthogonal.

error

$u_2 = \dfrac{v_2^{\perp}}{|v_2^{\perp}|}$

a) $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{3}}$ [illegible], $\vec{u}_2 = \frac{1}{\|\vec{v}_2\|}\vec{v}_2 = \frac{1}{\sqrt{3}}$ [illegible], $\vec{x} =$ [illegible]

$\text{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 = (1/\sqrt{3})\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} + (1/\sqrt{3})\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\-1/\sqrt{3}\end{pmatrix}$

$= \begin{pmatrix}1/3\\1/3\\1/3\end{pmatrix} + \begin{pmatrix}1/3\\1/3\\-1/3\end{pmatrix} = \begin{pmatrix}2/3\\2/3\\0\end{pmatrix}$ ans $= \begin{pmatrix}1/2\\1/2\\0\end{pmatrix}$

Leif anderson:
Solve $\begin{pmatrix}1 & 1\\1 & 1\\1 & -1\end{pmatrix}\vec{u} = \begin{pmatrix}1\\0\\0\end{pmatrix}$
by least squares. Then
$\text{proj}_V(\vec{x}) = \begin{pmatrix}1 & 1\\1 & 1\\1 & -1\end{pmatrix}\vec{u} = \begin{pmatrix}1/2\\1/2\\0\end{pmatrix}$

d) $A^T A \vec{x}^* = A^T \vec{b}$, $\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$

f) From the Gram-Schmidt process, we know that:

$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1$, $\vec{u}_2 = \dfrac{\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1}{\|\vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1\|} = \dfrac{\vec{v}_2 - \left(\frac{1}{\|\vec{v}_1\|}\vec{v}_1 \cdot \vec{v}_2\right)\frac{1}{\|\vec{v}_1\|}\vec{v}_1}{\left\|\vec{v}_2 - \left(\frac{1}{\|\vec{v}_1\|}\vec{v}_1 \cdot \vec{v}_2\right)\frac{1}{\|\vec{v}_1\|}\vec{v}_1\right\|}$

Let $V = \text{span}(\vec{u}_1, \vec{u}_2)$. Since $\vec{u}_1$ and $\vec{u}_2$ are linear combinations of $\vec{v}_1$ and $\vec{v}_2$

$\Rightarrow \vec{u}_1$ and $\vec{u}_2 \in \text{span}(\vec{v}_1, \vec{v}_2)$. $\vec{v}_1$ and $\vec{v}_2$ are independent and span $\vec{u}_1$ and $\vec{u}_2$

$\Rightarrow V = \text{span}(\vec{v}_1, \vec{v}_2) \Rightarrow \text{span}(\vec{u}_1, \vec{u}_2) = \text{Span}(\vec{v}_1, \vec{v}_2)$

$$A = \begin{pmatrix} 1 & 0 \\ 7 & 7 \\ 1 & 2 \end{pmatrix} \Rightarrow \text{Use Theorem 5.2.2 (pp. 207 textbook).}$$

$$\Rightarrow A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix}, \text{ where } \vec{v}_1 = \begin{pmatrix} 1 \\ 7 \\ 1 \end{pmatrix} \; \& \; \vec{v}_2 = \begin{pmatrix} 0 \\ 7 \\ 2 \end{pmatrix}$$

$$\Rightarrow \text{Then, } Q = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 \end{pmatrix} \; \& \; R = \begin{pmatrix} \|\vec{v}_1\| & \vec{u}_1 \cdot \vec{v}_1 \\ 0 & \|\vec{v}_2^{\perp}\| \end{pmatrix}$$

Note: R is upper triangular.

$\Rightarrow$ Use Theorem 5.2.1 (pp. 206 textbook):

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{1^2+7^2+1^2}}\begin{pmatrix} 1 \\ 7 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{51}}\begin{pmatrix} 1 \\ 7 \\ 1 \end{pmatrix} \qquad \|\vec{v}_1\| = \sqrt{51}$$

$$\vec{u}_1 \cdot \vec{v}_1 = \frac{1}{\sqrt{51}}\begin{pmatrix} 1 & 7 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 7 \\ 1 \end{pmatrix} = \frac{51}{\sqrt{51}} = \sqrt{51}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \begin{pmatrix} 0 \\ 7 \\ 2 \end{pmatrix} - \overbrace{\cancel{\frac{1}{\sqrt{51}}\begin{pmatrix} 1 & 7 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 7 \\ 2 \end{pmatrix}}}^{\sqrt{51}} \frac{1}{\sqrt{51}}\begin{pmatrix} 1 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp} = \frac{1}{\sqrt{1+0+1}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow Q = \begin{pmatrix} \frac{1}{\sqrt{51}} & -\frac{1}{\sqrt{2}} \\ \frac{7}{\sqrt{51}} & 0 \\ \frac{1}{\sqrt{51}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad R = \begin{pmatrix} \sqrt{51} & \sqrt{51} \\ 0 & \sqrt{2} \end{pmatrix}$$

$\Rightarrow$ Verification

$$\begin{pmatrix} \frac{1}{\sqrt{51}} & -\frac{1}{\sqrt{2}} \\ \frac{7}{\sqrt{51}} & 0 \\ \frac{1}{\sqrt{51}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \sqrt{51} & \sqrt{51} \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 7 & 7 \\ 1 & 2 \end{pmatrix}$$

See Theorem 5.3.4 (pp. 213 textbook); already proved in the book.

$\Rightarrow$ $T(\vec{x}) = AB\vec{x}$: show this preservs the lenght, i.e.: $\|AB\vec{x}\| = \|\vec{x}\|$.

$\Rightarrow$ $\|T(\vec{x})\| = \|A(B\vec{x})\| = \|B\vec{x}\| = \|\vec{x}\|$ ← Thus, AB is orthogonal.

because B is orthogonal (5.3.1).

because A is orthogonal and by 5.2.1 : $T(B\vec{x}) = A(B\vec{x}) = B\vec{x}$

c. $y = c_0 + c_1 x \longrightarrow$

$2 = c_0 + c_1 \cdot 0 \quad (0,2)$

$0 = c_0 + c_1 \quad (1,0)$

$1 = c_0 + 2c_1 \quad (2,1)$

$1 = c_0 + 3c_1 \quad (3,1)$

$$\rightarrow A\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \vec{y} \rightarrow \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}}_{\text{inconsistent system}}$$

Thm 5.4.2: If $\ker(A) = \{\vec{0}\}$, $A^T A$ is invertible and the least-squares solution c^* can be written as:

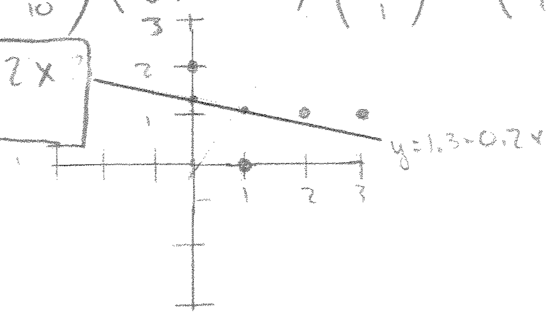
$$c^* = (A^T A)^{-1} A^T \vec{y}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \\ 0 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \ker A = \{\vec{0}\}$$

$$A^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \quad A^T A = \begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix} \quad (A^T A)^{-1} = \frac{1}{20}\begin{pmatrix} 14 & -6 \\ -6 & 4 \end{pmatrix} = \begin{pmatrix} \frac{7}{10} & \frac{-3}{10} \\ \frac{-3}{10} & \frac{2}{10} \end{pmatrix}$$

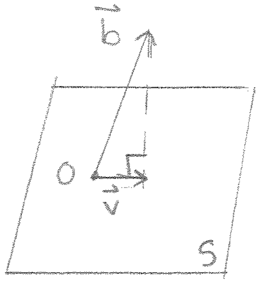
$$c^* = \begin{pmatrix} \frac{7}{10} & \frac{-3}{10} \\ \frac{-3}{10} & \frac{2}{10} \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{7}{10} & \frac{4}{10} & \frac{1}{10} & \frac{-2}{10} \\ \frac{-3}{10} & \frac{-1}{10} & \frac{1}{10} & \frac{3}{10} \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{13}{10} \\ \frac{-2}{10} \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

$$\boxed{y = 1.3 - 0.2x}$$



e. Near Point Theorem: For a vector $\vec{b}$ in $\mathbb{R}^n$, there is a vector $\vec{v}$ in subspace S which minimizes the distance between $\vec{b}$ and S. $\|\vec{b} - \vec{v}\| \leq \|\vec{b} - \vec{x}\|$ for all $\vec{x}$ in S. The vector $\vec{v}$ is the orthogonal projection of $\vec{b}$ onto S ($\text{proj}_S \vec{b}$).

$\vec{v} = \vec{b}$ when $\vec{b}$ is a vector in the subspace S.



h. Thm 5.3.3: A matrix A is orthogonal if its columns form an orthonormal basis of $\mathbb{R}^n$. If the columns of a matrix are independent, then A has an inverse A^{-1}.

$\|A\vec{x}\| = \|\vec{x}\|$ if A is orthogonal

$$\|A^{-1}\vec{x}\| = \|\underbrace{A(A^{-1}}_{AA^{-1} = I_n}\vec{x})\| = \|\vec{x}\|$$

$\therefore$ A has an inverse A^{-1} which is orthogonal.

3c.

Mike Shriere
2270 Gustafson
Midterm 2 Sample Problem, Group 3

Fit $c_0 + c_1 x$ to the data points $(0,2), (1,0), (2,1), (3,1)$ using least squares

$$S(c_0, c_1) = (2 - c_0)^2 + (0 - c_0 - c_1)^2 + (1 - c_0 - 2c_1)^2 + (1 - c_0 - 3c_1)^2$$

$$) = \frac{\partial S}{\partial c_0} = -2(2 - c_0) - 2(-c_0 - c_1) - 2(1 - c_0 - 2c_1) - 2(1 - c_0 - 3c_1)$$

$$) = \frac{\partial S}{\partial c_1} = -2(-c_0 - c_1) - 4(1 - c_0 - 2c_1) - 6(1 - c_0 - 3c_1)$$

$$0 = -4 + 2c_0 + 2c_0 + 2c_1 - 2 + 2c_0 + 4c_1 - 2 + 2c_0 + 6c_1$$
$$= -8 + 8c_0 + 12c_1$$
$$0 = +2c_0 + 2c_1 - 4 + 4c_0 + 8c_1 - 6 + 6c_0 + 18c_1$$
$$= 12c_0 + 28c_1 - 10$$

$$\begin{bmatrix} 8 & 12 \\ 12 & 28 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 12 & 8 \\ 12 & 28 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 6 & 14 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & 9 & 12 \\ 6 & 14 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & 9 & 12 \\ 0 & 5 & -7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & -7 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & 15 & 20 \\ 0 & 15 & -21 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & 0 & 41 \\ 0 & 15 & -21 \end{bmatrix}$$

$$10c_0 = 41 \qquad 15c_1 = -21$$
$$c_0 = \frac{41}{10} \qquad c_1 = \frac{-21}{15}$$