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for Problem 2

2-a Follows from the rank nullity theorem. See pg. 129.

2-b $p(x) = c_0 + c_1 x + c_2 x^2$

$p(1) = c_0 + c_1 + c_2$

$$\int_{-1}^{1} p(x)\,dx = 2c_0 + 0 + \frac{2c_2}{3} = c_0 + c_1 + c_2 \Rightarrow -c_0 + c_1 + \tfrac{1}{3}c_2 = 0$$

So $p(x) = (c_1 + \tfrac{1}{3}c_2) + c_1 x + c_2 x^2$

$$\left.\begin{array}{l} \dfrac{\partial p}{\partial c_1} = 1 + x \\[2ex] \dfrac{\partial p}{\partial c_2} = \dfrac{1}{3} + x^2 \end{array}\right\} \Rightarrow \text{basis} = \text{span}\left(1+x, \tfrac{1}{3} + x^2\right)$$

2-c

$$\left.\begin{array}{l} \dfrac{\partial s}{\partial a} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \\[3ex] \dfrac{\partial s}{\partial b} = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \end{array}\right\} \Rightarrow \text{basis} = \text{span}\left(\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}\right)$$

-d Let V be the space spanned by the orthonormal basis $\{\bar{v}_1, \ldots, \bar{v}_k\}$.

Consider

$$c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_k\bar{v}_k = \bar{0}$$

Take the dot product

$$(c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_k\bar{v}_k)\bar{v}_i = \bar{0}\cdot\bar{v}_i$$

Vector distribution yields

$$c_1\bar{v}_1\cdot\bar{v}_i + c_2\bar{v}_2\cdot\bar{v}_i + \cdots + c_i\bar{v}_i\cdot\bar{v}_i + \cdots c_k\bar{v}_k\cdot\bar{v}_i = c_1\cdot 0 + c_2\cdot 0 + \cdots + c_i + \cdots + c_k\cdot 0 = 0$$

This shows us that $c_i = 0 \Rightarrow c_1 = c_2 = \cdots = c_i = \cdots = c_k = 0$

$\therefore$ $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ are linearly independent $\blacksquare$

$\underline{\text{2-e}}$ $Col(I,1) = Col(rref(A), j_1) = E\,Col(A, j_1)$

$Col(I,2) = Col(rref(A), j_2) = E\,col(A, j_2)$

$\vdots \qquad \vdots \qquad \vdots$

$Col(I,K) = Col(rref(A), j_K) = E\,col(A, j_K)$

$$\sum_{i=1}^{k} c_i\,col(A, j_i) = \bar{0}$$

$$\sum c_i\,E\,col(A, j_i) = E\bar{0} = \bar{0} \Rightarrow \sum_{i=1}^{k} c_i\,col(I, j_i) = \bar{0} \Rightarrow \text{all } c_i = 0$$

$\boxed{\text{2-f}}$ Let $f(x) = 0\cdot 1 + 0\cdot x + 0\cdot(1-x) + 0\cdot e^x$

$f(x) \in V_1$ so V_1 contain 0. (1)

Furthermore V_1 is the span of $\{1, x, 1-x, e^x\}$ (2)

By (1) and (2) V_1 is a subspace of V

$f(0) = f(1) \Rightarrow a_0 + a_2 + a_3 = a_0 + a_1 + a_3 e \Rightarrow a_1 = a_2 + a_3 - a_3 e$

so $f(x) = (a_0 + a_2) + a_3(1-e)x + a_3 e^x$

S contains 0 because $(0+0) + 0(1-e)x + 0\cdot e^x$ is in S (3)

Let $g(x) = (a_0 + a_1) + a_3(1-e)x + a_3 e^x$

$h(x) = (b_0 + b_1) + b_3(1-e)x + b_3 e^x$

then $g(x) + h(x) = (a_0 + a_1 + b_0 + b_1) + (a_3 + b_3)(1-e)x + (a_3 + b_3)e^x$ is in S. (4)

$Kg(x) = (Ka_0 + Ka_1) + Ka_3(1-e)x + Ka_3 e^x$ is in S. (5)

(3), (4), and (5) show S is a subspace ■

$\boxed{2\text{-}g}$ Let V be the space of all m×n matrices.

Let $P = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}$ then $\dfrac{\partial P}{\partial a_{11}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 \end{bmatrix}$, $\dfrac{\partial P}{\partial a_{12}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & & 0 \end{bmatrix}$,

$\dfrac{\partial P}{\partial a_{21}} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & & \vdots \\ 0 & & \ddots & \vdots \\ \vdots & & & \\ 0 & \cdots & & 0 \end{bmatrix}, \ldots, \dfrac{\partial P}{\partial a_{mn}} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & 0 \; 0 \\ 0 & \cdots & & 0 \; 1 \end{bmatrix}$

This means $P = a_{11}\dfrac{\partial P}{\partial a_{11}} + a_{12}\dfrac{\partial P}{\partial a_{12}} + a_{21}\dfrac{\partial P}{\partial a_{21}} + \cdots + a_{mn}\dfrac{\partial P}{\partial a_{mn}}$

which implies the basis of $V = \operatorname{span}\left\{ \dfrac{\partial P}{\partial a_{11}}, \dfrac{\partial P}{\partial a_{12}}, \dfrac{\partial P}{\partial a_{21}}, \ldots, \dfrac{\partial P}{\partial a_{mn}} \right\}$ ∎

$\boxed{2\text{-}h}$ Let V be the vector space of all polynomials.

Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n = 0$

Repeatedly take the derivative of f w.r.t. x

$$\frac{df}{dx} = a_1 + 2a_2 + \cdots + n a_n x^{n-1}$$

$$\frac{d^2 f}{dx^2} = 2a_2 + 3! a_3 x + \cdots + n(n-1) a_n x^{n-2}$$

$$\vdots$$

$$\frac{d^n f}{dx^n} = n! a_n = 0 \implies a_n = 0$$

By back substitution we see $a_0 = a_1 = \cdots = a_n = 0$

$\therefore$ $1, x, \ldots, x^n$ are linearly independent ∎

2-i Let $\bar{v}_x \in \mathrm{Ker}(A)$

then $A\bar{v}_x = \bar{0}$

But $\bar{0}$ is dependent with all vectors so $A\bar{v}_1, A\bar{v}_2, A\bar{v}_3, A\bar{v}_x$ could not be independent even though $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_x$ are independent. Therefore $\mathrm{Ker}(A) = \{\bar{0}\}$ ∎

2-j We can see that $1-x$, $2x$, $(1-x)^2$ are linearly independent so we only need to verify that they span.

Suppose that another linearly independent vector exists then we would have four linearly independent vectors in a 3-space which is impossible.

2-k $\begin{bmatrix}1&0&0&0\\0&0&0&-1\end{bmatrix}, \begin{bmatrix}1&1&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&1&1&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1&1\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&1\\1&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\1&1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&1&1&0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&0&1&1\end{bmatrix}$

These 8 matrices are linearly independent by inspection. As such they only need span the space. Given that it's an 8 dimensional space they must span or they couldn't be linearly independent.

2-L S contains 0 because $0\cdot x + 0\cdot x^2 + 0\cdot x^3 + 0\cdot x^4 = 0$ is in S. (1)

Let $f(x) = a_0x + a_1x^2 + a_2x^3 + a_3x^4$

$g(x) = b_0x + b_1x^2 + b_2x^3 + b_3x^4$

then $f(x)+g(x) = (a_0+b_0)x + (a_1+b_1)x^2 + (a_2+b_2)x^3 + (a_3+b_3)x^4$ is in S. (2)

Finally $kf(x) = ka_0x + ka_1x^2 + ka_2x^3 + ka_3x^4$ is in S. (3)

(1), (2), & (3) show S is a subspace ∎